h-PRINCIPLE FOR STRATIFIED SPACES

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Abstract. We extend Gromov and Eliashberg-Mishachev's $h-$principle on manifolds to stratified spaces. This is done in both the sheaf-theoretic framework of Gromov and the smooth jets framework of Eliashberg-Mishachev. The generalization involves developing

(1) the notion of stratified continuous sheaves to extend Gromov's theory,
(2) the notion of smooth stratified bundles to extend Eliashberg-Mishachev's theory.

A new feature is the role played by homotopy fiber sheaves. We show, in particular, that stratumwise flexibility of stratified continuous sheaves along with flexibility of homotopy fiber sheaves furnishes the parametric h-principle. We extend the Eliashberg-Mishachev holonomic approximation theorem to stratified spaces. We also prove a stratified analog of the Smale-Hirsch immersion theorem.

Contents

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1. Introduction

Gromov developed the h-principle [Gro86] as a soft topological approach to finding solutions to partial differential relations, and this was refined subsequently by several others, notably Eliashberg-Mishachev [EM02]. These two references [Gro86, EM02] emphasize somewhat different points of view. While [EM02] uses the standard terminology of differential topology in terms of jets, [Gro86] uses a more abstract, formal sheaf-theoretic framework. The main applications of both these approaches is to solve partial differential relations on *smooth manifolds*. The aim of this paper is to extend the domain of applicability of the h-principle to *smooth stratified spaces* (cf. [GM88]).

There is one immediate difficulty that we face. Let X be a smooth stratified space. Then, any natural notion of a tangent bundle TX (cf. Definition 2.22) to X gives a structure that is not a bundle in the usual sense of a topological bundle. This leads us to the notion of *stratified bundles* (Definition 2.14) which consist of bundles along strata of X with appropriate gluing conditions across strata. It is precisely this "gluing data" that distinguishes the manifold framework from the stratified spaces framework. A prototypical example of a stratified bundle to keep in mind is that of $P : M \longrightarrow M/G$, where M is a smooth manifold, and G is a compact Lie group with a not necessarily free smooth action on M. Then M admits a stratification by orbit type, where the strata $M_{(H)}$ are indexed by closed subgroups $H \leq G$, consisting of points with isotropy group conjugate to H. This stratification descends to a stratification of M/G, and $P : M \to M/G$ becomes a stratified fiber bundle. Let $x \in M$, G_x be the isotropy group, and $[x] = P(x)$. Then the fiber of P over $[x]$ is G/G_x.

We found the sheaf-theoretic formalism of Gromov [Gro86, Ch. 2] more convenient to address the algebraic topology issues. We refer the reader to [Gro86, Ch. 2] for details on sheaves of quasitopological spaces, i.e. continuous sheaves. Note however that the sheaf of sections of a stratified fiber bundle E over a stratified space X forms something more involved than just a sheaf over a topological space, as one may restrict E to any stratum-closure $\overline{L} \subset X$ and take sections thereof. We therefore extend the sheaf-theoretic formalism of Gromov to *stratified sites* (Definition 3.6) and *stratified sheaves* (Definition 3.8) over these. In cases of interest in this paper, a stratified sheaf typically assigns a topological space to an open subset U of a stratum closure $\overline{L}$. Here, L ranges over strata of X. Following Gromov [Gro86], we call such a stratified sheaf a *stratified continuous sheaf*. Thus, a stratified continuous

sheaf is a collection of continuous sheaves $\{\mathcal{F}_{\overline{L}}\}$, one for every stratum L of X. In this sense, stratified continuous sheaves are analogous to constructible sheaves.

Next, for every pair $S < L$, the data of a stratified continuous sheaf gives a restriction map

$$\mathrm{res}_S^L : i_{\overline{S}}^* \mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{S}},$$

where $i_{\overline{S}}^* \mathcal{F}_{\overline{L}}$ denotes the pullback of $\mathcal{F}_{\overline{L}}$ to $\overline{S}$. A homotopy theoretic construction that arises naturally is that of the homotopy fiber (hofib):

$$\overline{\mathcal{H}}_S^L = \mathrm{hofib}(\mathrm{res}_S^L : i_{\overline{S}}^* \mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{S}}).$$

Set $\mathcal{H}_S^L = \overline{\mathcal{H}}_S^L | S$, the restriction of $\overline{\mathcal{H}}_S^L$ to the stratum S of $\overline{S} \subset X$. Thus,

$$\mathcal{H}_S^L = \mathrm{hofib}(\mathrm{res}_S^L : \mathcal{F}_{\overline{L}} \to \mathcal{F}_S).$$

We shall refer to $\overline{\mathcal{H}}_S^L$ and $\mathcal{H}_S^L$ as the closed and open homotopy fiber sheaves respectively for the pair of strata $S < L$. One of the aims of this paper is to find conditions on the homotopy fiber sheaves that guarantee the $h-$principle for the stratified sheaf $\mathcal{F}$.

The sheaf of sections of a bundle comes naturally equipped with the compact-open topology, and therefore constitutes a continuous sheaf. A key difference between Gromov's h-principle [Gro86, Section 2.2] and the present paper stems from the fact that a stratified sheaf is a collection of sheaves, and not a single sheaf. The difference already shows up when the base space is a simplex equipped with its natural stratification. This change in setup was motivated by a question due to Sullivan. In fact, this paper and its companion [MS22] were born in part by trying to address the following two questions due to Sullivan [Sul77] and Gromov [Gro86]. After discussing smooth forms on simplices in [Sul77], Sullivan suggests the following Example/Problem/test-question.

Question 1.1 (stratified spaces)**.** [Sul77, p. 298] *"In the $\cdots$ cell-space abstraction we didn't require that cells be contractible. Thus these notions can be extended to stratified sets-thought of inductively as obtained by attaching manifolds with boundary with a careful statement about the geometry of the attaching map.*

It would be interesting to carry this out in detail-the basic idea being that a form should have values only on multivectors tangent to the strata."

We interpret Question 1.1 as follows:

Question 1.2. *Provide an inductive description of forms over stratified spaces.*

Another source of inspiration for this paper comes from the following question due to Gromov.

Question 1.3. [Gro86, p. 343] *Can one define singular symplectic (sub) varieties?*

In this paper, we address Question 1.2, by providing an inductive description of sheaves of smooth forms over stratified spaces. In fact, we set up the more general framework of a stratified bundle over a stratified space, and provide an inductive description of the sheaf of sections of a stratified bundle over a stratified space. We postpone a full treatment of Question 1.3 to the companion paper [MS22], but develop a general conceptual framework in this paper.

Having fixed the framework in terms of stratified sheaves over stratified spaces, the main aim of the paper then boils down to extending some of the basic notions

introduced by Gromov in [Gro86, Ch. 2] to the stratified context, and proving the stratified h-principle using these generalizations. Two crucial concepts were important in [Gro86, Ch. 2] for a sheaf $\mathcal{F}$ (of topological spaces) over a manifold V:

(1) Flexibility of $\mathcal{F}$: This means that for every $K \subset K' \subset V$, $\mathcal{F}(K') \to \mathcal{F}(K)$ is a (Serre) fibration.
(2) $\mathrm{Diff}(V)-$invariance of $\mathcal{F}$: This means that the action of the pseudogroup $\mathrm{Diff}(V)$ of partially defined diffeomorphisms of V lifts to an action on $\mathcal{F}$.

Flexibility of sheaves is generalized to flexibility of stratified sheaves by demanding two kinds of conditions:

(1) *Stratumwise flexibility:* For every stratum $S < X$, the restricted sheaf $\mathcal{F}|S$ (assigning $\mathcal{F}(U)$ only to open subsets U of S) is flexible as a sheaf over the manifold S. Using Gromov's work [Gro86], this hypothesis allows us to conclude the $h-$principle for the restrictions of $\mathcal{F}$ to *open strata*.
(2) *Flexibility across strata*: For every stratum $S < X$, the open homotopy fiber sheaf $\mathcal{H}^L_S$ is flexible on the open stratum S. As pointed out before, for any pair of strata $S < L$ of X, there exist closed and open homotopy fiber sheaves $\overline{\mathcal{H}}^L_S$ and $\mathcal{H}^L_S$ respectively. Of these, the open homotopy fiber sheaf $\mathcal{H}^L_S$ is a more tractable object and is defined on the open (manifold) stratum S. *$\mathcal{H}^L_S$ is the key new player introduced in this paper in the context of stratified spaces.* No analog exists in the smooth manifold context. It is the sheaf $\mathcal{H}^L_S$ that encodes "gluing data" across the pair of strata S, L.

We establish (Theorem 4.42) that if (1) and (2) holds, then $\mathcal{F}$ satisfies the $h-$principle.

The condition of $\mathrm{Diff}(V)-$invariance in [Gro86] is generalized to invariance of the stratified sheaf $\mathcal{F}$ under stratified diffeomorphisms (Definition 3.14). Thus, $\mathcal{F}$ is StratDiff $-$invariant if for every pair of strata $S < L$ of X, $\mathcal{F}|S$, $i_S^*\mathcal{F}_{\overline{L}}$ and the restriction map res^L_S are naturally $\mathrm{Diff}(S)-$invariant.

Terminology, examples and non-examples: What we have called "stratified bundles" have their origins in work of Thom [Tho69]. Mather [Mat12, p. 500] defines the notion of a "Thom map". In this paper a stratified bundle map is a Thom map satisfying an extra hypothesis (Definition 2.14, condition (iii)) making the notion closer to a bundle.
1) The main class of examples of stratified bundles over stratified spaces, as mentioned earlier, consists of $P : M \to M/G$, where M is a smooth manifold, and G is a compact Lie group. This includes symplectic reductions [SL91] [MMeO$^+$07, Theorems 1.4.2, 2.4.2]. In fact, if G_1, G_2 admit commuting actions on M, e.g. if the G_1 action is on the left, and the G_2 action is on the right, then there exists a stratified bundle $P_1 : G_1\backslash M \to G_1\backslash M/G_2$, where $G_1\backslash M$ itself is allowed to be a stratified space. Thus, there are natural examples of stratified bundles of the form $P : X \to X/G$, where X is itself a stratified space.
2) *Caveat:* The reader should be warned that in the context of complex analytic spaces, a genuine topological bundle over a stratified complex analytic space with holomorphic total space, and fiber a complex manifold is sometimes referred to as a stratified bundle (cf. [For10]). We use stratified bundle in a much more general sense than this. In particular, the fibers over different strata need not be homeomorphic

in our context.
3) Let $D \subset V$ be a singular divisor in a smooth complex variety V. Let $N_\varepsilon(D)$ be a regular neighborhood, and $\partial N_\varepsilon(D)$ its boundary. Let $r : N_\varepsilon(D) \to D$ be the retraction map to the divisor. Then the restriction $r|\partial N_\varepsilon(D) : \partial N_\varepsilon(D) \to D$ is *not* an example of a stratified bundle in our sense. This is because lower dimensional strata in D have higher dimensional fibers under $r|\partial N_\varepsilon(D) : \partial N_\varepsilon(D) \to D$. For a stratified bundle in our sense, the opposite happens: for instance, lower dimensional strata in M/G have lower dimensional fibers under $P : M \to M/G$.

1.1. **Statement of results.** We are now in a position to state the first main theorem of our paper (referring to Theorem 4.42 for a more precise statement).

Theorem 1.4. *Let $\mathcal{F} = \{\mathcal{F}_{\overline{L}} : L < X \text{ stratum}\}$ be a stratified continuous sheaf over a stratified space X, such that $\mathcal{F}$ is stratumwise flexible, i.e. $\mathcal{F}_{\overline{L}}|L$ for each $L < X$ is flexible. Further, suppose that $\mathcal{F}$ is infinitesimally flexible across strata, i.e. each open homotopy fiber sheaf $\mathcal{H}_S^L$ is flexible. Then $\mathcal{F}$ satisfies the parametric $h-$principle.*

Gromov deduces the h-principle from the homotopy-theoretic condition of flexibility for sheaves, notably over manifolds [Gro86, p.76]. Theorem 1.4 is the stratified analog of Gromov's theorem: the underlying space is replaced by a stratified space, and sheaves are replaced by stratified sheaves.

The following theorem of Gromov connects flexibility and microflexibility [Gro86, Ch. 2.2].

Theorem 1.5. [Gro86, p. 78] *Let $Y = V \times \mathbb{R}$ and let $\Pi : Y \to V$ denote the projection onto the first factor. Let $\mathcal{F}$ be a microflexible continuous sheaf over Y invariant under* $\mathrm{Diff}(V, \Pi)$. *Then the restriction $\mathcal{F}|V \times \{0\}$ is a flexible sheaf over $V (= V \times \{0\})$.*

Let $\mathcal{F}$ be a microflexible $\mathrm{Diff}(V)-$*invariant continuous sheaf over a manifold V. Then the restriction to an arbitrary piecewise smooth polyhedron $K \subset V$ of positive codimension, $\mathcal{F}|K$, is a flexible sheaf over K.*

The stratified analog of Theorem 1.5 is then furnished by the following (see Theorems 4.47 and 4.49):

Theorem 1.6. *Let $\mathcal{F} = \{\mathcal{F}_{\overline{L}} : L < X \text{ stratum}\}$ be a stratified continuous sheaf over a stratified space X, such that $\mathcal{F}$ is* StratDiff*-invariant. Further, suppose for each stratum $L < X$, $\mathcal{F}_{\overline{L}}|L$ is microflexible and for each pair of strata $S < L < X$, $\mathcal{H}_S^L$ is microflexible. Then the restriction $\mathcal{F}|K$ to a stratified subspace $K \subset X$ of stratumwise positive codimension satisfies the parametric $h-$principle.*

In Section 5, we address Sullivan's Question 1.2 by developing a flag-like structure for jets on stratified spaces. The main results are given by Propositions 5.2, 5.8, and 5.14. These results give a stratified analog of the sheaf of formal $r-$jets in [EM02, EM01] (see Definition 5.13 giving the corresponding sheaf $\mathrm{Strat}\,\mathcal{J}^r$). In a sense, Section 5 interpolates between the algebraic topology of Section 4 and the differential topology of Section 6.

In Section 6, we return to the differential topological setting of jets and jet bundles as a concrete example to which the above sheaf-theoretic theorems may be applied. Let $p : E \to X$ be a smooth stratified bundle over a smooth stratified space X. Then the sheaves of sections of $p : E \to X$ and their jets come with natural

control conditions. Let $\mathcal{F}$ denote the stratified sheaf of controlled sections of E over X. Then we have the following (see Theorem 6.4).

Theorem 1.7. *$\mathcal{F}$ is flexible, in particular it satisfies the parametric $h-$principle.*

A caveat is in order. The continuous sheaf $\mathcal{F}$ is strictly smaller than the sheaf of *all* sections. It is rather easy to see that the sheaf of all sections satisfies the parametric $h-$principle. Theorem 1.7 says that this continues to hold in the presence of control conditions. In fact, $\mathcal{F}$ can be identified with the sheaf of holonomic stratified $r-$jets (see Definition 5.13 for the sheaf $\operatorname{Strat}\mathcal{J}^r$ of formal stratified $r-$jets), and hence Theorem 1.7 is also true for the sheaf of holonomic stratified jets.

We establish the following stratified holonomic approximation theorem (see Theorem 6.21), generalizing Eliashberg-Mishachev's holonomic approximation theorem [EM01, Theorem 1.2.1] for manifolds (Theorem 6.11).

Theorem 1.8. *Let X be an abstractly stratified space equipped with a compatible metric, $E \to X$ be a stratified bundle, and $K \subset X$ be a relatively compact stratified subspace of positive codimension. Let $f \in \operatorname{Strat}\mathcal{J}^r_E(\operatorname{Op} K)$ be a C^r-regular formal section. Then for arbitrarily small $\varepsilon > 0$, $\delta > 0$, there exist a stratified diffeomorphism $h : X \to X$ with*

$$||h - \mathrm{Id}||_{C^0} < \delta,$$

and a stratified holonomic section $\widetilde{f} \in \operatorname{Strat}\mathcal{J}^r_E(\operatorname{Op} K)$ such that

1. *the image $h(K)$ is contained in the domain of definition of the section f,*
2. *$||\widetilde{f} - f|\operatorname{Op} h(K)||_{C^0} < \varepsilon$.*
3. *$\widetilde{f}, f|\operatorname{Op} h(K)$ are normally ε C^r-close.*

We should point out that neither Theorem 1.7, nor Theorem 1.8 follow from the relative holonomic approximation theorem [EM02, Theorem 3.2.1]. The basic issue can be illustrated in the simple case of a pair of strata $S < L$. Let E_S, E_L denote the bundles over S, L. In order to prove either of these theorems, we need to consider *extensions* of a jet (formal or holonomic) of E_S over S to a jet over a germ of a neighborhood of S in $\overline{L}$. This echoes the fact mentioned earlier, that in the stratified sheaf context, the open homotopy fiber $\mathcal{H}^L_S$ is a new player in the game. Alternately, the extension may be thought of as "gluing data" that allows us to go between jets over S and jets over L.

A host of applications in the manifold context have been enumerated by Eliashberg-Mishachev. All have potential generalizations to the stratified context in the light of Theorem 1.8. We give an application of our techniques at the end of Section 6.2 by showing that stratified immersions of positive codimension between stratified spaces satisfy the h-principle (Theorem 6.24): this is the stratified analog of the Smale-Hirsch theorem [EM02, Chapter 8.2].

1.2. **Outline of the paper.** The aim of Section 2 is to set up the context of the h-principle for stratified spaces. This section is in the spirit of [EM02], except that the technology is for stratified spaces in place of manifolds. We start by describing the setup of stratified spaces following [GM88, Mat12]. Stratified spaces are recalled in Section 2.1 and stratified maps in Section 2.2. We define stratified bundles and related notions in Section 2.3, where the basic fact we prove is the local structure of bundle maps (Lemma 2.15 and Corollary 2.17). It is well-known that locally a stratified space looks like a product $\mathbb{R}^n \times cA$ of Euclidean space with a cone cA

on a compact stratified space A. Lemma 2.15 and Corollary 2.17 upgrade this to a statement about the local structure of a stratified bundle over a stratified space. We then proceed in Section 2.4 to define stratified jets, jet bundles, and formal and holonomic sections in the stratified context.

While Section 2 ends by setting up the context of the h-principle for stratified spaces by describing jet bundles and their sections and is in the spirit of [EM02], Section 3 has more of an algebraic topology flavor, and is in the spirit of [Gro86]. Here, we look at sheaves over stratified spaces. The crucial notion of a *stratified sheaf* over a stratified space is introduced in Section 3.2. Flexibility conditions are introduced in this context in Section 3.2. A principal condition used in [Gro86] is Diff-invariance of sheaves. In Section 3.3, we describe the stratified analog, StratDiff $-$invariance, in the context of stratified spaces.

In Section 4, we prove one of the main theorems of the paper, Theorem 4.42 (or Theorem 1.4 above), establishing the parametric $h-$principle for stratified sheaves over stratified spaces. The main idea or mnemonic may be summarized as follows: flexibility (a precursor to the h-principle) normal to strata plus flexibility tangential to strata furnishes the $h-$principle for stratified sheaves over stratified spaces. In a sense, this is in the spirit of Goresky-Macpherson's fundamental theorem on Morse data [GM88] on stratified spaces where total Morse data can be recovered from normal Morse data and tangential Morse data. On the way, we establish Theorem 4.19, spelling out the connection between flexibility and the h-principle in the context of stratified sheaves. A tool we use in Section 4 is Milnor's theory of microbundles. This allows us to simplify Gromov's formalism from [Gro86, Chapter 2].

In Section 4.5, we establish the connection between flexibility and microflexibility of stratified sheaves (see Theorem 4.47). It follows (see Theorem 4.49) that the restriction of microflexible StratDiff $-$invariant sheaves to positive codimension stratified subspaces satisfies the parametric $h-$principle.

Section 5 is devoted to using microbundles and developing a homotopy model of the Gromov diagonal normal sheaf $\mathcal{F}^*$ for a sheaf $\mathcal{F}$ of controlled sections of a stratified bundle $P : E \to X$. In so doing, we answer Sullivan's Question 1.2 by developing a formalism of flag-like sets. The description is hybrid in nature. There is a tangential component given by sections along manifold strata S and there is a normal component given by sections over the link of S in X. Since the link A of a stratum S is itself a stratified space, the restriction of $P : E \to X$ to $P : P^{-1}(A) \to A$ is again a stratified bundle of lesser complexity; hence an inductive description.

We return to jets and jet bundles of stratified bundles over stratified spaces in Section 6. Theorem 6.4 establishes that the sheaf of sections of a stratified jet bundle satisfies the parametric $h-$principle. We also prove the stratified analog of Eliashberg-Mishachev's holonomic approximation theorem in Theorem 6.21. As an application of Theorem 1.4, we establish a stratified Smale-Hirsch theorem: stratified immersions of positive codimension between stratified spaces satisfy the h-principle (Theorem 6.24).

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2. Smoothly stratified objects and maps

2.1. **Smoothly stratified spaces.**

Definition 2.1. *Let X be a locally compact second countable metric space and let $(I, \leq)$ be a partially ordered set. An* I-decomposition *of X is a locally finite collection $\{S_\alpha\}_{\alpha \in I}$ of disjoint locally closed subsets of X such that*

1. *S_α is a topological manifold for all $\alpha \in I$*
2. *$X = \bigcup_{\alpha \in I} S_\alpha$*
3. *$S_\alpha \cap \overline{S_\beta} \neq \emptyset \iff S_\alpha \subseteq \overline{S_\beta} \iff \alpha \leq \beta$.*

If X is an I-decomposed space as above, we shall call S_α, $\alpha \in I$, the *strata* of X, and denote by Σ the collection of strata of X indexed by I. We shall use $\alpha \leq \beta$ and $S_\alpha \leq S_\beta$ interchangeably, partially ordering Σ instead. If $S_\alpha \leq S_\beta$ and $S_\alpha \neq S_\beta$ we shall write $\alpha < \beta$ (or $S_\alpha < S_\beta$). Note that $S_\alpha < S_\beta$ is equivalent to saying that S_α lies in the boundary

$$\partial S_\beta := \overline{S_\beta} \setminus S_\beta$$

of S_β. For any stratum $S \in \Sigma$ we define the *depth* of S to be

$$\operatorname{depth}(S) := \sup\{n : S_i \in \Sigma \text{ such that } S < S_1 < \cdots < S_{n-1}\}.$$

Similarly, define the *height* of S to be

$$\operatorname{height}(S) := \sup\{n : S_i \in \Sigma \text{ such that } S > S_1 > \cdots > S_{n-1}\}.$$

We shall moreover define the *depth* and *dimension* of X respectively as

$$\operatorname{depth}(X) := \sup_{\alpha \in I} \operatorname{depth}(S_\alpha), \quad \dim X := \sup_{\alpha \in I} \dim(S_\alpha)$$

Definition 2.2. *Let (S, L) be a pair of smooth (not necessarily properly) embedded submanifolds of a smooth manifold M such that $S \subset \overline{L}$.*

1. *The pair (S, L) is said to satisfy* Whitney condition (a) *if for any sequence $\{x_n\} \subset L$ converging to $x \in S$ such that the sequence of tangent planes $T_{x_n} L$ converge to a plane $\tau \subset T_x M$, the inclusion $T_x S \subset \tau$ holds.*
2. *The pair (S, L) is said to satisfy* Whitney condition (b) *if the following holds.*
   Let $\{x_n\} \subset L$, $\{y_n\} \subset S$ be any pair of sequences both converging to $x \in S$ such that the tangent planes $T_{x_n} L$ converge to some plane $\tau \in T_x M$. Further suppose that the secants $\overline{x_n y_n}$ converge to a line $\ell \in T_x M$. Then $\ell \subset \tau$.

The notions of convergence of planes and lines mentioned above are defined locally by choosing a coordinate chart around x in M, such that the chart contains a tail of the sequences $\{x_n\}, \{y_n\}$. It is straightforward to check that the property of the pair (S, L) satisfying either of the above conditions is independent of the chosen coordinate chart. See [Mat12] for a coordinate-free restatement of condition (b). Note that condition (b) implies condition (a), since given any sequence $\{x_n\} \subset L$ with $T_{x_n} L \to \tau$ and a line $\ell \subset T_x S$, defined in a local chart $(U, x) \cong (\mathbb{R}^n, 0)$ around x, one can construct a pair of sequences $\{x_n\} \subset L$ and $\{y_n\} \subset S$ such that the secants $\overline{x_n y_n}$ converge to $\ell \in T_x M$. Now as $T_{x_n} L \to \tau$, we must have $\ell \subset \tau$ by hypothesis of satisfying condition (b). Therefore, $\ell \subset \tau$. But $\ell \subset T_x S$ was arbitrary, so we conclude $T_x S \subset \tau$. Therefore, condition (a) holds.

Definition 2.3. *A* Whitney stratified subset *of a smooth manifold M is a subset $X \subset M$ with an I-decomposition Σ such that every stratum in Σ is a smoothly embedded submanifold of M, and any pair of strata (S, L) in Σ such that $S < L$ satisfies the Whitney condition (b).*

Thom and Mather showed that every Whitney stratified subset of a smooth manifold has a canonical local model akin to manifolds being locally modeled by Euclidean spaces. This gives us an intrinsic definition of a topological Whitney stratified set. The cone on a topological space A is denoted as cA.

Definition 2.4. [Fri19] *A* CS set *is an I-decomposed space (X, Σ) such that for any stratum $S \in \Sigma$ the following holds:*
For any point $x \in S$ there exists an open neighborhood U of x in X, a chart V around x in S, and a stratified space (A, Σ_A) called the link *of x, such that there is a stratum-preserving homeomorphism $\varphi : V \times cA \to U$ where U is given the induced stratification from X.*

Theorem 2.5. [Mat12] *Every Whitney stratified subset of a smooth manifold is a CS set.*

Given an I-decomposed space (X, Σ), a *tubular neighborhood system* or simply, a *tube system* $\mathcal{N}$ on Σ is a collection of triples $(N_\alpha, \pi_\alpha, S_\alpha)_{\alpha \in I}$ consisting of (for every $\alpha \in I$) an open neighborhood N_α of the stratum S_α in X, called the *tubular neighborhood* of the stratum, a retraction $\pi_\alpha : N_\alpha \to S_\alpha$, called the *tubular projection*, and a continuous function $\rho_\alpha : N_\alpha \to [0, \infty)$ such that $\rho_\alpha^{-1}(0) = S_\alpha$: ρ_α is called the *radial function*.

We now define an *abstract stratification* in the sense of Mather [Mat12]. It provides a notion of a smooth stratification on an I-decomposed space (X, Σ) independent of the ambient space it is embedded in. This is analogous to the abstract definition of a smooth manifold using a smooth atlas rather than via an embedding.

Definition 2.6. *An I-decomposed space (X, Σ) equipped with a tube system $\mathcal{N}$ on Σ, denoted by the triple $(X, \Sigma, \mathcal{N})$, defines an* abstractly stratified space *if the following holds*

1. *Each stratum $S_\alpha \in \Sigma$ is a smooth manifold.*
2. *For any pair $\alpha, \beta \in I$ of indices such that $\alpha < \beta$, we set $N_{\alpha\beta} = N_\alpha \cap S_\beta$ and the restrictions of π_α and ρ_α to $N_{\alpha\beta}$ are denoted by $\pi_{\alpha\beta}$ and $\rho_{\alpha\beta}$ respectively. The map $(\pi_{\alpha\beta}, \rho_{\alpha\beta}) : N_{\alpha\beta} \to S_\alpha \times (0, \infty)$ is a submersion.*
3. *For all triples $\alpha, \beta, \gamma \in I$ of indices with $\alpha < \beta < \gamma$, the* π-control condition *$\pi_{\alpha\beta}\pi_{\beta\gamma}(x) = \pi_{\alpha\gamma}(x)$ and the* ρ-control condition *$\rho_{\alpha\beta}\pi_{\beta\gamma}(x) = \rho_{\alpha\gamma}(x)$ are satisfied whenever $x \in N_{\beta\gamma} \cap N_{\alpha\gamma} \cap \pi_{\beta\gamma}^{-1}(N_{\alpha\beta})$.*

For simplicity of notation, we will often denote the tubular neighborhood of a stratum $S \in \Sigma$ in an abstractly stratified space $(X, \Sigma, \mathcal{N})$ by N_S and the associated tubular function and radial function will be denoted by π_S and ρ_S. For two strata $S, L \in \Sigma$, $S < L$, the tubular neighborhood of S in L will be defined as $N_{SL} := N_S \cap L$ and the restrictions of π_S and ρ_S to N_{SL} will be denoted by π_{SL} and ρ_{SL} respectively. Let

$$N_S^\varepsilon := \{x \in N_S : \rho_S(x) < \varepsilon(\pi_S(x))\} \subset N_S,$$

where $\varepsilon : S \to (0, \infty)$ is a *smooth positive continuous function*. We shall often use the shorthand $\varepsilon > 0$ if there is no scope of confusion. N_S shall usually mean N_S^1.

Here, we use ε to denote a function on S defining a tubular neighborhood of S. However, for most applications, we shall need to consider the function ε only on relatively compact subsets of S, where it may be taken to be a small constant, hence this notation.

Two tube systems $\mathcal{N} = (N_S, \pi_S, \rho_S)_{S\in\Sigma}$ and $\mathcal{N}' = (N'_S, \pi'_S, \rho'_S)_{S\in\Sigma}$ on an I-decomposed space (X, Σ) shall be declared *equivalent* if for any strata $S \in \Sigma$, there exists an open neighborhood $S \subset N''_S \subset N_S \cap N'_S$ such that $\pi_S|N''_S = \pi'_S|N''_S$ and $\rho_S|N''_S = \rho'_S|N''_S$. If $(X, \Sigma_X, \mathcal{N}_X)$ and $(Y, \Sigma_Y, \mathcal{N}_Y)$ are two abstractly stratified spaces and $f : X \to Y$ is a stratum-preserving homeomorphism such that the pulled back tube system $f^*\mathcal{N}_Y = (f^*N_S, f^{-1} \circ \pi_S \circ f, \rho_S \circ f)_{S\in\Sigma_Y}$ is equivalent to $\mathcal{N}_X$, then f is said to be an *isomorphism* between X and Y.

Examples of stratified spaces include manifolds with corners. A product $X \times Y$ of stratified spaces X, Y is naturally stratified with strata consisting of products of strata of X and Y, however there is no canonically defined abstract stratification in general. For instance, consider $I \times I$ where $I = [0, 1]$ is stratified as a manifold with boundary. However, if one of X or Y is a manifold, $X \times Y$ does have a canonical abstract stratification.

Theorem 2.7. [Mat12] *Any Whitney stratified subset $(X, \Sigma) \subset M$ admits a tubular neighborhood system consisting of (not necessarily properly) embedded tubular neighborhoods $\nu(S)$, one for each stratum $S \in \Sigma$ in M. Further, there exists a projection $\pi_S : \nu(S) \to S$ and radial function $\rho_S : \nu(S) \to [0, \infty)$ such that $N_S = \nu(S) \cap X$. The restrictions of π_S and ρ_S to N_S furnish a tube system $\mathcal{N} = (N_S, \pi_S, \rho_S)_{S\in\Sigma}$ which makes $(X, \Sigma, \mathcal{N})$ an abstractly stratified space.*

The following theorem is a version of the Whitney embedding theorem for abstractly stratified spaces, essentially saying that every abstractly stratified space of dimension n can be embedded in $\mathbb{R}^N$ for $N \geq 2n + 1$ as a Whitney stratified space, and any two such embeddings are isotopic if $N \geq 2n + 2$.

Theorem 2.8. [Nat80] *Let $(X, \Sigma, \mathcal{N})$ be an abstractly stratified space with* $\dim X = n$*. Then for any $N \geq 2n + 1$ there is a* realization *of X in $\mathbb{R}^N$, i.e. there exists an embedding $\iota : X \to \mathbb{R}^N$ such that $X' = \iota(X)$ is a Whitney stratified subset of $\mathbb{R}^N$ with a stratification $\Sigma' = \{\iota(S) : S \in \Sigma\}$ and a tube system $\mathcal{N}' = (N_S, \pi_S, \rho_S)$ as in Theorem 2.7, such that*

$$\iota : (X, \Sigma, \mathcal{N}) \to (X, \Sigma', \mathcal{N}')$$

is an isomorphism of abstractly stratified spaces. Moreover if $N \geq 2n + 2$ any two such embeddings $\iota_0, \iota_1 : X \to \mathbb{R}^N$ are isotopic in the following sense: there is a realization $H : X \times I \to \mathbb{R}^N$ such that $H(x, t) = (H_t(x), t)$ where $H_t : X \to \mathbb{R}^N$ is a realization for all $0 \leq t \leq 1$ and $H_0 = \iota_0, H_1 = \iota_1$.

Theorem 2.9. [Gor78] *Any abstractly stratified space admits a triangulation by smoothly embedded simplices compatible with the filtration by stratum-closures*

2.2. **Stratified maps.**

Definition 2.10. *A map $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$ of I-decomposed spaces is said to be a* stratum-preserving map *if for any $S \in \Sigma_X$, there is a unique $L \in \Sigma_Y$ such that $f(S) \subset L$. Equivalently, for every stratum $L \in \Sigma_Y$, $f^{-1}(L)$ is a disjoint union of strata of Σ_X.*

If $(X, \Sigma_X, \mathcal{N}_X)$ and $(Y, \Sigma_Y, \mathcal{N}_Y)$ are abstractly stratified spaces, then a stratum-preserving map $f : X \to Y$ of the underlying I-decomposed spaces is said to be a

controlled map *if for any stratum $S \in \Sigma_X$ and the corresponding unique stratum $L \in \Sigma_Y$ such that $f(S) \subset L$, the following conditions are satisfied*

(1) *$f|S : S \to L$ is a smooth map.*
(2) *There exists $\varepsilon > 0$ such that $f(N_S^\varepsilon) \subset N_L$.*
(3) *The* π-control *condition*

$$f(\pi_S(x)) = \pi_L(f(x))$$

and the ρ-control *condition*

$$\rho_S(x) = \rho_L(f(x))$$

hold for all $x \in N_S^\varepsilon$.

If all the conditions above except the ρ-control condition is satisfied, f is said to be a weakly controlled map*.*

Definition 2.11. *A controlled map $f : (X, \Sigma_X, \mathcal{N}_X) \to (Y, \Sigma_Y, \mathcal{N}_Y)$ is a* stratified submersion *if for any stratum $L \in \Sigma_Y$ and any component $S \in \Sigma_X$ of $f^{-1}(L)$, $f|S : S \to L$ is a submersion.*

2.3. **Stratified bundles.**

Definition 2.12. [Mat12] *Let $(X, \Sigma, \mathcal{N})$ be an abstractly stratified space. A* stratified vector field *η on X is a collection $\{\eta_S : S \in \Sigma\}$ where for each stratum $S \in \Sigma$, η_S is a smooth vector field on S. The stratified vector field η will be called a* controlled vector field *if for any pair $S, L \in \Sigma$ of strata with $S < L$, there exists some $\varepsilon > 0$ such that for any $x \in N_S^\varepsilon \cap L$, the following conditions hold:*

(1) *$\eta_L \rho_{SL}(x) = 0$.*
(2) *$(\pi_{SL})_* \eta_L(x) = \eta_S(\pi_{SL}(x))$.*

If we simply drop the first condition, we obtain a weakly controlled vector field*.*

Thus, a controlled vector field in the higher dimensional stratum L is parallel to the lower dimensional stratum S, i.e. it does not change along the radial direction ρ_{SL}. This is ensured by the first condition above. It also projects 'isomorphically' to the vector field in the lower dimensional stratum S. This is ensured by the second condition above. A weakly controlled vector field in the higher dimensional stratum L is not necessarily parallel to the lower dimensional stratum S, i.e. it is allowed to have a radial component; however, if the radial component is subtracted from a weakly controlled vector field, we obtain a controlled vector field.

We now define higher-dimensional controlled distributions. These will be useful in defining stratified fiber bundles below. Note that we drop the ρ-control condition in this case.

Definition 2.13. *Let $(X, \Sigma, \mathcal{N})$ be an abstractly stratified space. A* stratified distribution *D on X is a collection $\{D_S : S \in \Sigma\}$ where for each stratum $S \in \Sigma$, D_S is a smooth subbundle of the tangent bundle TS of S. The distribution D will be called a* weakly controlled distribution *if for any pair $S, L \in \Sigma$ of strata with $S < L$, there exists some $\varepsilon > 0$ such that for any $x \in T_S^\varepsilon \cap L$,*

$$(\pi_{SL})_* D_L(x) = D_S(\pi_{SL}(x)).$$

Note that the dimensions of D_S, D_L may differ for $S \neq L$. For the next definition, we shall need the local structure of neighborhoods N_S of strata S. By Thom's first isotopy lemma, N_S is a fiber bundle over S with fiber cA, where A denotes the link of S in X.

Definition 2.14. *A triple (E, X, p) consisting of*

(1) *An abstractly stratified space $(X, \Sigma, \mathcal{N})$, called the* base space*,*
(2) *An abstractly stratified space $(E, \widetilde{\Sigma}, \widetilde{\mathcal{N}})$, called the* total space*,*
(3) *A weakly controlled map $p : E \to X$ called the* bundle projection*,*

will be called a stratified fiber bundle *if*

(i) *For every stratum $\widetilde{S} \in \widetilde{\Sigma}$ and the corresponding unique stratum $S \in \Sigma$ such that $p(\widetilde{S}) \subseteq S$, the restriction $p : \widetilde{S} \to S$ is a smooth fiber bundle, and*
(ii) *The stratified distribution* $\ker dp := \{\ker d(p|_{\widetilde{S}}) : \widetilde{S} \in \widetilde{\Sigma}\}$ *on E is weakly controlled.*
(iii) *Let $p|N_{\widetilde{S}} : N_{\widetilde{S}} \to N_S$ denote the restriction of p to a neighborhood $N_{\widetilde{S}}$ of $\widetilde{S}$. Let B, A denote the links of $\widetilde{S}, S$ respectively, so that $N_{\widetilde{S}}$ (resp. N_S) is a bundle over $\widetilde{S}$ (resp. S) with fiber cB (resp. cA). Identify $\widetilde{S}$ with the zero-section of $N_{\widetilde{S}}$. We demand that $(p|N_{\widetilde{S}})^{-1}(S) = \widetilde{S}$.*

Given a stratified fiber bundle (E, X, p), a (weakly) controlled section *of p is a (weakly) controlled map $s : X \to E$ such that $p \circ s = \mathrm{id}$.*

We should point out that a stratified bundle (E, X, p) as defined above is necessarily locally trivial over strata by Thom's second isotopy lemma, originally formulated in [Tho69] (for a detailed proof, see [Mat12, Proposition 11.2]). Hence, we can think of a stratified bundle as a collection $\{(E_S, S, p) : S \in \Sigma\}$, where each $p : E_S \to S$ is a genuine topological bundle over the stratum S with fiber a stratified space. The conditions of Definition 2.14 ensure that these bundles patch together consistently. Condition (ii) is known as Thom's condition (a_p) in literature [Mat12, Section 11]. Condition (iii) forces the restriction $p|cB$ of p to a $cB-$fiber of $N(\widetilde{S})$ to land in a $cA-$ fiber of $N(\widetilde{S})$ with the additional condition that the pre-image of the cone-point c_A of cA is exactly the cone-point c_B of cB. This will be useful in Corollary 2.18 below.

We shall sometimes use the suggestive notation $p : E \to X$ for a stratified fiber bundle. The following lemma and its consequences (Corollary 2.17 and Corollary 2.18) give the *local structure* of stratified fiber bundles.

Lemma 2.15. *Let (E, X, p) be a stratified fiber bundle. For any point $\widetilde{x} \in E$ with $p(\widetilde{x}) = x$, there is an open neighborhood V of $\widetilde{x}$ in E, and U of x in X, equipped with the respective induced stratifications, such that $p(V) = U$, and*

(1) *There exist abstractly stratified spaces $(A, \Sigma_A, \mathcal{N}_A)$, $(B, \Sigma_B, \mathcal{N}_B)$ and isomorphisms of abstractly stratified spaces $\psi : V \to cB \times \mathbb{R}^n$ and $\varphi : U \to cA \times \mathbb{R}^m$ for some $n \geq m$,*
(2) *There exists a map $f : cB \times \mathbb{R}^n \to cA \times \mathbb{R}^m$ which factors as $f = (g, \mathrm{proj})$ where $\mathrm{proj} : \mathbb{R}^n \to \mathbb{R}^m$ denotes the projection to the first m coordinates,*

making the following diagram commute:

$$\begin{array}{ccc} V & \longrightarrow & cB \times \mathbb{R}^n \\ \downarrow & & \downarrow \\ U & \longrightarrow & cA \times \mathbb{R}^m \end{array}$$

Proof. Suppose $\widetilde{S}$ is the unique stratum of E containing $\widetilde{x}$. Let S be the unique stratum of X containing x. So $p(\widetilde{S}) \subseteq S$. Let $\widetilde{\pi}$ be the tubular projection associated

to $\widetilde{S}$ in E and let π be the tubular projection associated to S in X. Since $p|\widetilde{S} : \widetilde{S} \to S$ is a fiber bundle, we can choose charts $\widetilde{O} \cong \mathbb{R}^n$ around $\widetilde{x}$ in $\widetilde{S}$ and $O \cong \mathbb{R}^m$ around x in S such that $p : \widetilde{O} \to O$ is equivalent to the projection $\mathrm{proj} : \mathbb{R}^n \to \mathbb{R}^m$ with respect to local coordinates. Let $V = \widetilde{\pi}^{-1}(\widetilde{O}) \cap \mathrm{cl}(N^{\varepsilon}_{\widetilde{S}})$ and $U = \pi^{-1}(O) \cap \mathrm{cl}(N^{\varepsilon}_{S})$ for some appropriate $\varepsilon > 0$ (here $\mathrm{cl}(-)$ denotes closure). Observe that $\pi : U \to O$ is a proper stratified submersion. Choose coordinate vector fields $\partial_1, \cdots, \partial_m$ on O corresponding to local coordinates $t_1, \cdots, t_m$. By [Mat12, Proposition 9.1] there exist controlled vector fields $\eta_1, \cdots, \eta_m$ on U which commute stratumwise such that $\pi_* \eta_i = \partial_i$ for all $1 \leq i \leq m$. Let $F = \pi^{-1}(x) \cap \mathrm{cl}(N^{\varepsilon}_{S})$. Let Φ_i be the local 1-parameter family of stratum-preserving homeomorphisms on U generated by η_i, for $0 \leq i \leq m$. For any $u \in U$, there exists a unique $v \in F$ and unique $(t_1, \cdots, t_m) \in O$, such that $(\Phi_1^{t_1} \circ \Phi_2^{t_2} \cdots \circ \Phi_m^{t_m})(v) = u$. This gives an inverse homeomorphism $h : U \to F \times O$ defined by

$$h(u) = ((\Phi_m^{-t_m} \circ \cdots \circ \Phi_2^{-t_2} \circ \Phi_1^{-t_1})(u), t_1, \cdots, t_m),$$

where $\pi_S(u) = (t_1, \cdots, t_m)$, so that $t_1, \cdots, t_m$ are (implicitly) functions of u.

Now consider the commutative square

$$\begin{array}{ccc} V & \xrightarrow{\widetilde{\pi}} & \widetilde{O} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ U & \xrightarrow{\pi} & O \end{array}$$

This gives us a map to the fibered product $(\widetilde{\pi}, p) : V \to \widetilde{O} \times_O U$. Note that since $\ker dp$ is a weakly controlled distribution on E by hypothesis, $\widetilde{\pi}_* \ker dp_v = \ker dp_{\widetilde{\pi}(v)}$, for any $v \in V$, i.e. $d\widetilde{\pi}$ restricts to a surjection $d\widetilde{\pi} : \ker dp_v \to \ker dp_{\widetilde{\pi}(v)}$. We now use a fact from linear algebra:

Claim 2.16. *Let $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4$ be vector spaces occurring in the following commutative diagram*

$$\begin{array}{ccc} \mathbf{W}_1 & \xrightarrow{f} & \mathbf{W}_2 \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow \\ \mathbf{W}_3 & \xrightarrow{g} & \mathbf{W}_4 \end{array}$$

where f, g, p, q are all surjective linear maps. If f restricts to a surjection $f : \ker p \to \ker q$ then the induced map to the fibered product $(f, p) : \mathbf{W}_1 \to \mathbf{W}_2 \times_{\mathbf{W}_4} \mathbf{W}_3$ is surjective.

Proof of Claim 2.16. For any $u \in \mathbf{W}_2$ and $v \in \mathbf{W}_3$ such that $q(u) = g(v)$, choose lifts $\widetilde{u}, \widetilde{v} \in \mathbf{W}_1$ such that $f(\widetilde{u}) = u$ and $p(\widetilde{v}) = v$ by surjectivity of f and p, respectively. Then observe that $(q \circ f)(\widetilde{u} - \widetilde{v}) = q(u) - (q \circ f)(\widetilde{v}) = q(u) - (g \circ p)(\widetilde{v}) = q(u) - g(v) = 0$ by commutativity of the diagram. Therefore, $f(\widetilde{u} - \widetilde{v}) \in \ker q$. As $f : \ker p \to \ker q$ is surjective, there must be some $k \in \ker p$ such that $f(\widetilde{u} - \widetilde{v}) = f(k)$. Therefore there must also be some $\ell \in \ker f$ such that $\widetilde{u} - \widetilde{v} = k + \ell$. Let $w = \widetilde{u} - \ell = \widetilde{v} + k \in \mathbf{W}_1$. This is the desired element. □

We now return to the proof of Lemma 2.15. Note that $\widetilde{O} \times_O U$ is an abstractly stratified space. Claim 2.16 implies $(\widetilde{\pi}, p) : V \to \widetilde{O} \times_O U$ is a stratumwise submersion. Therefore, there exist controlled vector fields $\widetilde{\eta}_i$ on V *over* η_i on U for $1 \leq i \leq m$, see [Mat12, Proposition 11.5] (we pause here to record a warning that controlled

vector fields *over* controlled vector fields are not controlled vector fields in the usual sense of the word, see [Mat12, Section 11] for a careful discussion). In particular, $\widetilde{\pi}_*\widetilde{\eta}_i = \widetilde{\partial}_i$ for all $1 \leq i \leq m$, where $\widetilde{\partial}_1, \cdots, \widetilde{\partial}_m$ are the first m coordinate vector fields on $\widetilde{O}$ obtained as lifts of $\partial_1, \cdots, \partial_m$ by the projection $p : \widetilde{O} \to O$. Consider the rest of the coordinate vector fields $\widetilde{\partial}_{m+1}, \cdots, \widetilde{\partial}_n$ on $\widetilde{O}$ and let $\widetilde{\eta}_{m+1}, \cdots, \widetilde{\eta}_n$ be controlled vector fields on V such that $\widetilde{\pi}_*\widetilde{\eta}_i = \widetilde{\partial}_i$ for $m+1 \leq i \leq n$.

Let $\widetilde{\Phi}_i$ denote the local 1-parameter family of stratum-preserving homeomorphisms of V generated by $\widetilde{\eta}_i$ for $1 \leq i \leq n$. We obtain, as before, two homeomorphisms $\widetilde{h}_1 : V \to \widetilde{F} \times \widetilde{O}$ given by $\widetilde{h}_1(v) = (\widetilde{\Phi}_n^{-t_n} \circ \cdots \circ \widetilde{\Phi}_1^{-t_1}(v), t_1, \cdots, t_n)$ and $\widetilde{h}_2 : V \to F' \times O$ given by $\widetilde{h}_2(v) = (\widetilde{\Phi}_m^{-t_m} \circ \cdots \circ \widetilde{\Phi}_1^{-t_1}(v), t_1, \cdots, t_m)$ by considering the flow generated by all of $\widetilde{\eta}_1, \cdots, \widetilde{\eta}_n$ in the first case, and the flow generated by the first m of these, namely $\widetilde{\eta}_1, \cdots, \widetilde{\eta}_m$, in the second case.

Here $\widetilde{F} = \widetilde{\pi}^{-1}(\widetilde{x}) \cap N_{\widetilde{S}}^{\varepsilon}$ and $F' = (p \circ \widetilde{\pi})^{-1}(x) \cap N_{\widetilde{S}}^{\varepsilon}$. Consider the map

$$\phi : \widetilde{F} \times \widetilde{O} \to F' \times O$$
$$\phi(z, t_1, \cdots, t_n) = (\Phi_{m+1}^{t_{m+1}} \circ \cdots \circ \Phi_n^{t_n}(z), t_{m+1}, \cdots, t_n)$$

Then the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\widetilde{h}_1} & \widetilde{F} \times \widetilde{O} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi} \\ V & \xrightarrow{\widetilde{h}_2} & F' \times O \end{array}$$

Since $\widetilde{\eta}_1, \cdots, \widetilde{\eta}_m$ are controlled vector fields on V *over* $\eta_1, \cdots, \eta_m$ on U, by [Mat12, Proposition 11.6] there is also a commutative diagram as follows

$$\begin{array}{ccc} V & \xrightarrow{\widetilde{h}_1} & F' \times O \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (p,\mathrm{id})} \\ U & \xrightarrow{h} & F \times O \end{array}$$

By combining the two commutative diagrams above, we obtain (up to change of coordinates) an equivalence of $p : V \to U$ with a map $\widetilde{F} \times \widetilde{O} \to F \times O$ defined by $(z, \mathbf{t}) \mapsto (g_{\mathbf{t}}(z), p(\mathbf{t}))$. Now, observe that the map $\widetilde{F} \times \widetilde{O} \to F \times \widetilde{O}$, $(z, \mathbf{t}) \mapsto (g_{\mathbf{t}}(z), \mathbf{t})$ is also a stratified fiber bundle. So we can lift coordinate vector fields $\widetilde{\partial}_1, \cdots, \widetilde{\partial}_n$ on $\widetilde{O}$ to controlled vector fields on $F \times \widetilde{O}$ and from there to controlled vector fields on $\widetilde{F} \times \widetilde{O}$ *over* the aforementioned controlled vector fields. Once again, using [Mat12, Proposition 11.6], we obtain a commutative diagram as follows

$$\begin{array}{ccc} \widetilde{F} \times \widetilde{O} & \xrightarrow{\cong} & \widetilde{F} \times \widetilde{O} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (g_{\mathbf{0}},p)} \\ F \times \widetilde{O} & = & F \times \widetilde{O} \end{array}$$

where the diagram is compatible with projections of each of the terms to $\widetilde{O}$. Therefore, by conjugating by the isomorphism on the top horizontal arrow we obtain an equivalence of $\widetilde{F} \times \widetilde{O} \to F$, $(z, \mathbf{t}) \mapsto g_{\mathbf{t}}(z)$ with $\widetilde{F} \times \widetilde{O} \to F$, $(z, \mathbf{t}) \mapsto g_{\mathbf{0}}(z)$. This

shows that $p : V \to U$ is equivalent (up to reparametrization) to a genuine product

$$g_{\mathbf{0}} \times p : \widetilde{F} \times \widetilde{O} \to F \times O$$

By an application of Thom's first isotopy lemma [Mat12, Proposition 11.1], we identify $F \cong cA$ and $\widetilde{F} \cong cB$ where $A = \pi^{-1}(x) \cap \rho^{-1}(r)$ and $B = \widetilde{\pi}^{-1}(\widetilde{x}) \cap \widetilde{\rho}^{-1}(r')$ for some sufficiently small $r, r' > 0$, with the induced abstract stratification from E and X respectively. The lemma follows. □

We refine the conclusions of Lemma 2.15 by elaborating on the structure of the map $p_c : cB \to cA$ between the conical factors in the local trivialization

$$(V, U, p|_V) \cong (cB \times \mathbb{R}^n, cA \times \mathbb{R}^m, p_c \times \mathrm{proj})$$

of a stratified fiber bundle (E, X, p) furnished by Lemma 2.15

Corollary 2.17. *$p_c : cB \to cA$ is a stratified fiber bundle.*

Proof. We know

$$p_c \times \mathrm{proj} : cB \times \mathbb{R}^n \to cA \times \mathbb{R}^m$$

is a stratified fiber bundle, since it is equivalent to the stratified fiber bundle $p|V : V \to U$ by a pair of isomorphisms of abstractly stratified spaces. Then for any stratum S of cA and any stratum L of cB such that $f(L) \subset S$, the restriction $p_c \times \mathrm{proj} : L \times \mathbb{R}^n \to S \times \mathbb{R}^m$ is a smooth fiber bundle. In particular, $p_c|L : L \to S$ is a surjective smooth submersion.

We can arrange p_c to be a proper map by choosing $r, r' > 0$ appropriately in the last part of Lemma 2.15. Therefore, $p_c|L$ is a proper surjective submersion and hence a smooth fiber bundle by Ehresmann's fibration theorem. Therefore, $p_c : cB \to cA$ is a weakly controlled map which is a stratumwise smooth fiber bundle. Moreover, it is straightforward to check that $\ker dp_c$ is a controlled distribution on cB since $\ker d(p_c \times \mathrm{proj})$ is a controlled distribution on $cB \times \mathbb{R}^n$ by hypothesis. This establishes that (cB, cA, p_c) satisfies all the hypotheses in Definition 2.14 and is therefore a stratified fiber bundle. □

Corollary 2.18. *The map $p_c : cB \to cA$ is equivalent to the cone on a map $p_\ell : B \to A$ between the links, by a pair of isomorphisms of the abstractly stratified spaces cA and cB. That is, there exists a commutative diagram of the form*

$$\begin{array}{ccc} cB & \xrightarrow{\cong} & cB \\ \downarrow{\scriptstyle p_c} & & \downarrow{\scriptstyle c(p_\ell)} \\ cA & \xrightarrow{\cong} & cA \end{array}$$

Proof. For concreteness, let $cA = A \times [0,1)/A \times \{0\}$ and $cB = B \times [0,1)/B \times \{0\}$, and let us indicate the cone points as $\{c_A\}$ and $\{c_B\}$ respectively. Identify the $[0,1)$ factor in each with radial co-ordinates on cA, cB, using the radial functions ρ_A and ρ_B, respectively. Let $\rho_A : cA \to [0,1)$ denote the projection onto its radial co-ordinate, and let $\Phi = \rho_A \circ p_c$.

Then $\Phi : cB \to [0,1)$ is a stratified fiber bundle with compact fibers, where the base $[0,1)$ has exactly two strata $\{0\}$ and $(0,1)$. Let c_B denote the cone-point of B. Then condition (iii) of Definition 2.14 ensures that $\Phi : cB \setminus \{c_B\} \to (0,1)$ is a stratified fiber bundle where the base is a single stratum, and the fibers are compact. By Thom's first isotopy lemma [Mat12, Proposition 11.1], $\Phi : cB \setminus \{c_B\} \to (0,1)$ is a product fibration, i.e. $cB \setminus \{c_B\}$ is isomorphic to $B \times (0,1)$ as abstractly stratified

spaces, by an isomorphism which preserves the projection to $(0,1)$. Reparametrizing the radial co-ordinates furnishes the conclusion. □

Let $(A, \Sigma_A, \mathcal{N}_A)$ and $(B, \Sigma_B, \mathcal{N}_B)$ be abstractly stratified spaces. The same argument as in Lemma 2.15 and Corollary 2.18 can be used to establish lifting of *stratified homotopies* $H : A \times I \to B$ i.e. a homotopy where, for every stratum $S \in \Sigma_A$, there exists a unique stratum $L \in \Sigma_B$ such that $H(I \times S) \subset I \times L$.

Proposition 2.19. *Let (E, B, p) be a stratified bundle, and $H : A \times [0,1] \to B$ be a stratified homotopy. Let $h_0 := H|A \times \{0\}$ and let $\widetilde{h}_0 : A \to E$ be a lift of h_0. Then there exists a lift $\widetilde{H} : A \times [0,1] \to E$ such that $\widetilde{H}$ is a stratified homotopy, $\widetilde{H}|A \times \{0\} = \widetilde{h}_0$ and $\widetilde{H}$ covers H, i.e. $\widetilde{H} \circ p = H$.*

Proof. We modify the homotopy by enlarging $[0,1]$ slightly to $(-\varepsilon, 1+\varepsilon)$ and defining $H : A \times (-\varepsilon, 1+\varepsilon) \to B$ by declaring H to be constant on $A \times (-\varepsilon, 0]$ and $A \times [1, 1+\varepsilon)$. It is now possible to choose $\varepsilon > 0$ such that H is a stratified mapping, where $A \times (-\varepsilon, 1+\varepsilon)$ is stratified by $S \times (-\varepsilon, 1+\varepsilon)$; $S \in \Sigma_A$ being the strata of A.

Let $H^*E \subset A \times (-\varepsilon, 1+\varepsilon) \times E$ denote the pullback of E over $A \times (-\varepsilon, 1+\varepsilon)$. Since H is a stratified map, H^*E is a stratified bundle over $A \times (-\varepsilon, 1+\varepsilon)$. By projecting first to $A \times (-\varepsilon, 1+\varepsilon)$ and then to $(-\varepsilon, 1+\varepsilon)$ as in Corollary 2.18, we obtain a commutative diagram

$$\begin{array}{ccc} H^*E & \xrightarrow{\cong} & E_0 \times (-\varepsilon, 1+\varepsilon) \\ \downarrow & & \downarrow \\ A \times (-\varepsilon, 1+\varepsilon) & = & A \times (-\varepsilon, 1+\varepsilon) \end{array}$$

Where $E_0 = h_0^*E$ is the pullback of the stratified fiber bundle (E, B, p) over A under $h_0 : A \to B$. The map $\widetilde{h}_0 : A \to E$ induces a map to the fibered product $H^*(\widetilde{h}_0) : A \to H^*E$.

Let $\Phi : H^*E \to E_0 \times (-\varepsilon, 1+\varepsilon)$ denote the isomorphism above. Then, the product homotopy with coordinates changed by Φ, i.e.

$$H = \Phi^{-1} \circ (\Phi \circ H^*(\widetilde{h}_0), t)$$

gives the required lift. □

A slight generalization of stratified fiber bundles is sometimes useful:

Definition 2.20. *A* stratumwise bundle $P : E \to B$ *consists of*

1. *an abstractly stratified space E-the* total space*,*
2. *an abstractly stratified space B-the* base space*,*
3. *a stratum-preserving map P, such that for every stratum S of B,*

$$P|P^{-1}(S) : P^{-1}(S) \to S$$

is a topological fiber bundle, with fiber a stratified space F_S.

Example 2.21. *A product of stratified spaces is a stratumwise bundle, but not necessarily a stratified fiber bundle.*

Definition 2.22. *Let $(X, \Sigma, \mathcal{N})$ be an abstract C^∞-stratified space. By Theorem 2.8 there is a realization $X' \subset \mathbb{R}^N$ such that $(X', \Sigma', \mathcal{N}')$ is an abstract stratified set, where $\mathcal{N}'$ is induced from a tubular neighborhood system $(\nu(S), \pi_S, \rho_S)_{S \in \Sigma'}$ on the Whitney stratified set $X' \subset \mathbb{R}^n$. We define the* **tangent bundle** *TX to be the*

union $\bigcup_{S\in\Sigma'} TS \subset T\mathbb{R}^N$ *of tangent bundles to each strata of* X'*. This inherits a topology from* $T\mathbb{R}^N = \mathbb{R}^{2N}$ *and an* I*-decomposition* $\Sigma^{(1)} = \{TS_\alpha : S_\alpha \in \Sigma'\}_{\alpha\in I}$*.*

Let $p : TX \to X$ *be the* **projection** *obtained by restricting the projection* $T\mathbb{R}^N \to \mathbb{R}^N$ *to* TX *and composing with the inverse of the realization homeomorphism* $X \to X'$*. Define*

$$N_\alpha^{(1)} := T\nu(S_\alpha) \cap TX$$

to be the tube around the stratum TS_α *of* TX*, i.e.* $N_\alpha^{(1)}$ *is the intersection of* TX *with the tangent bundle to the tubular neighborhood* $\nu(S_\alpha)$*.*

The associated tubular projection $\pi_\alpha^{(1)} := d\pi_\alpha$ *is defined by the restriction of the differential*

$$d\pi_\alpha : T\nu(S_\alpha) \to TS_\alpha$$

to $N_\alpha^{(1)}$*.*

Finally, consider the differential of the radial function $d\rho_\alpha : T\nu(S_\alpha) \to \mathbb{R}$ *as a map to the fibers of* $T[0,\infty) \cong [0,\infty) \times \mathbb{R}$*. Then we define the radial function associated to the stratum* TS_α *as*

$$\rho_\alpha^{(1)} := \rho_\alpha \circ p + (d\rho_\alpha)^2.$$

We denote the tube system defined by these functions as $\mathcal{N}^{(1)} = (N_\alpha^{(1)}, \pi_\alpha^{(1)}, \rho_\alpha^{(1)})_{\alpha\in I}$*.*

Lemma 2.23. *The triple* $(TX, \Sigma^{(1)}, \mathcal{N}^{(1)})$ *is an abstract* C^∞*-stratified space.*

Proof. For any pair of indices $\alpha, \beta \in I$ with $\alpha < \beta$, $N_{\alpha\beta} = N_\alpha \cap S_\beta$ is a submanifold of S_β. Hence it inherits a C^∞-structure. Consider $\pi_{\alpha\beta} : N_{\alpha\beta} \to S_\alpha$ and $\rho_{\alpha\beta} : N_{\alpha\beta} \to (0,\infty)$ – both C^∞-maps. If $\pi_{\alpha\beta}^{(1)}$ and $\rho_{\alpha\beta}^{(1)}$ denote the restrictions of $\pi_\alpha^{(1)}$ and $\rho_\alpha^{(1)}$ to

$$N_{\alpha\beta}^{(1)} = N_\alpha^{(1)} \cap TS_\beta = TN_\alpha \cap TS_\beta = T(N_\alpha \cap S_\beta) = TN_{\alpha\beta},$$

then observe that $\pi_{\alpha\beta}^{(1)} = d\pi_{\alpha\beta}$ and $\rho_{\alpha\beta}^{(1)} = \rho_{\alpha\beta} \circ p + (d\rho_{\alpha\beta})^2$.

We know that for any triple of indices $\alpha, \beta, \gamma \in I$, the following control conditions hold:

$$\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma} \tag{1}$$

$$\rho_{\alpha\beta} \circ \pi_{\beta\gamma} = \rho_{\alpha\gamma} \tag{2}$$

Differentiating and using the chain rule on 1 and 2, we have

$$d\pi_{\alpha\beta} \circ d\pi_{\beta\gamma} = d\pi_{\alpha\gamma} \tag{3}$$

$$d\rho_{\alpha\beta} \circ d\pi_{\beta\gamma} = d\rho_{\alpha\gamma} \tag{4}$$

whenever both sides of the equations are defined. Equation 3 implies that

$$\pi_{\alpha\beta}^{(1)} \circ \pi_{\beta\gamma}^{(1)} = \pi_{\alpha\gamma}^{(1)},$$

hence $(TX, \Sigma^{(1)}, \mathcal{N}^{(1)})$ has π-control.

From Equation 4 we also obtain

$$(d\rho_{\alpha\beta})^2 \circ d\pi_{\beta\gamma} = (d\rho_{\alpha\gamma})^2 \tag{5}$$

Since $p \circ d\pi_{\beta\gamma} = \pi_{\beta\gamma}$, we see that Equation 2 also implies

$$(\rho_{\alpha\beta} \circ p) \circ d\pi_{\beta\gamma} = \rho_{\alpha\gamma} \circ p \tag{6}$$

Adding Equations 5 and 6, we see that $\rho_{\alpha\beta}^{(1)} \circ \pi_{\beta\gamma}^{(1)} = \rho_{\alpha\beta}^{(1)}$. This is the desired ρ-control.

Note that $\rho^{(1)}_\alpha(x,v) = 0$ if and only if $\rho_\alpha(x) = d\rho_\alpha(x,v) = 0$. Since $\rho_\alpha(x) = 0$, $x \in S_\alpha$. Next write $v = u + w \in T\mathbb{R}^N$ where u is the orthogonal projection of v to T_xS_α under a fiberwise inner product defined on $\nu(S_\alpha)$. Then

$$0 = d\rho_\alpha(x,v) = d\rho_\alpha(x,u) + d\rho_\alpha(x,w).$$

But $d\rho_\alpha(x,u) = 0$ since $\rho_\alpha \equiv 0$ on S_α. This forces $d\rho_\alpha(x,w) = 0$. Since ρ_α is the radial function on $\nu(S_\alpha)$, $d\rho_\alpha$ is strictly increasing in any direction orthogonal to S_α, forcing $w = 0$. Hence $v \in T_xS_\alpha$. Therefore $(x,v) \in TS_\alpha$, i.e., $(\rho^{(1)}_\alpha)^{-1}(0) = TS_\alpha$. Finally it is straightforward to check that $(\pi^{(1)}_{\alpha\beta}, \rho^{(1)}_{\alpha\beta}) : N^{(1)}_{\alpha\beta} \to S_\beta \times (0,\infty)$ is a submersion.

We have checked that $\pi^{(1)}$ and $\rho^{(1)}$ are valid projection and radial functions, and $(TX, \Sigma^{(1)}, \mathcal{N}^{(1)})$ satisfies both the control conditions. The lemma follows. □

Lemma 2.24. *(Weakly) controlled sections of the projection $p : TX \to X$ are (weakly) controlled vector fields on X (cf. Definition 2.12).*

Proof. For concreteness, we prove the lemma for controlled sections and controlled vector fields. The same proof works for weakly controlled sections and weakly controlled vector fields. Let $\eta : X \to TX$ be a controlled section of p. Then for any stratum $S \subset X$, $\eta|S : S \to TS$ is a C^∞-section of the tangent bundle of S; let us denote this vector field as η_S. Then $\{\eta_S : S \in \Sigma\}$ is a stratified vector field on X. Note that since η is π-controlled, it follows that for any pair of strata $S, L \subset X$ with $S < L$, we have $\pi^{(1)}_{SL}(\eta(x)) = \eta(\pi_{SL}(x))$. Hence $(\pi_{SL})_*(\eta_L(x)) = \eta_S(\pi_{SL}(x))$. Moreover η is ρ-controlled, hence $\rho^{(1)}_{SL}(\eta(x)) = \rho_{SL}(x)$. Now $\rho^{(1)}_{SL}(\eta(x)) = \rho_{SL}(x) + d\rho_{SL}(\eta(x))^2$, therefore $d\rho_{SL}(\eta(x)) = 0$. Equivalently, $(\eta_L)_*\rho_{SL} = 0$. This verifies that $\{\eta_S : S \in \Sigma\}$ is indeed a controlled vector field on X. □

Proposition 2.25. *Let $(X, \Sigma_X, \mathcal{N}_X)$ and $(Y, \Sigma_Y, \mathcal{N}_Y)$ be abstract stratified spaces and $f : X \to Y$ be a controlled (resp. weakly controlled) map. Then the stratum-wise differential induces a controlled (resp. weakly controlled) map $df : TX \to TY$.*

Proof. Suppose $\alpha, \beta \in I$ is a pair of indices such that $\alpha < \beta$, and $S_\alpha, S_\beta \in \Sigma_X$ be the corresponding pair of strata of X. Let $L_\alpha, L_\beta \in \Sigma_Y$ be the unique strata of Y such that $f(S_\alpha) \subset L_\alpha$ and $f(S_\beta) \subset L_\beta$. Let us denote the tube system on X associated to the pair of strata (S_α, S_β) by $(N^X_{\alpha\beta}, \pi^X_{\alpha\beta}, \rho^X_{\alpha\beta})$ and similarly the tube system on Y associated to the pair of strata (L_α, L_β) by $(N^Y_{\alpha\beta}, \pi^Y_{\alpha\beta}, \rho^Y_{\alpha\beta})$. As f is a controlled mapping, we have

$$f \circ \pi^X_{\alpha\beta} = \pi^Y_{\alpha\beta} \circ f \tag{7}$$

$$\rho^X_{\alpha\beta} = \rho^Y_{\alpha\beta} \circ f \tag{8}$$

on $N^X_{\alpha\beta} \cap f^{-1}(N^Y_{\alpha\beta})$. Differentiating Equation 7, it is immediate that $df \circ (\pi^X)^{(1)}_{\alpha\beta} = (\pi^Y)^{(1)}_{\alpha\beta} \circ df$. Differentiating Equation 8 we obtain

$$d\rho^X_{\alpha\beta} = d\rho^Y_{\alpha\beta} \circ df \tag{9}$$

Since $(p \circ df)(x,v) = f(x)$ for all $(x,v) \in TX$, we can rewrite Equation 8 as

$$\rho^X_{\alpha\beta} \circ p = \rho^Y_{\alpha\beta} \circ p \circ df \tag{10}$$

Squaring both sides of Equation 9 and adding to Equation 10 gives $(\rho^X)^{(1)}_{\alpha\beta} = (\rho^Y)^{(1)}_{\alpha\beta} \circ df$. This verifies that $df : (TX, \Sigma^{(1)}_X, \mathcal{N}^{(1)}_X) \to (TY, \Sigma^{(1)}_Y, \mathcal{N}^{(1)}_Y)$ is a controlled

map. It is clear from the proof that if f is only weakly controlled, df is also weakly controlled. □

2.4. **Stratified jets.** Let $(X, \Sigma, \mathcal{N})$ be an abstract stratified space. Then the tangent bundle $(TX, \Sigma^{(1)}, \mathcal{N}^{(1)})$ is also an abstractly stratified space by Lemma 2.23. We can now iterate this construction to define the **k-fold iterated tangent bundle** $T^{(k)}X$. However, the radial functions become rather unwieldy to work with. We redefine the control structure on $T^{(k)}X$ as follows:

Definition 2.26 (Iterated tangent bundle)**.** *Let $(X, \Sigma, \mathcal{N})$ be an abstract stratified space and we choose a realization $X' \subset \mathbb{R}^N$ such that $(X', \Sigma', \mathcal{N}')$ is an abstract stratified set. Here, $\mathcal{N}'$ is induced from a tubular neighborhood system $(\nu(S), \pi_S, \rho_S)_{S\in\Sigma'}$ on the Whitney stratified set $X' \subset \mathbb{R}^N$.*

Let $T^{(k)}X$ be the union $\bigcup_{S\in\Sigma'} T^{(k)}S \subset T^{(k)}\mathbb{R}^N$ of the k-fold iterated tangent bundles to each stratum of X'. Then $T^{(k)}X$ inherits a topology from $T^{(k)}\mathbb{R}^N = \mathbb{R}^{2^k N}$ and an I-decomposition $\Sigma^{(k)} = \{T^{(k)}S_\alpha : S_\alpha \in \Sigma'\}_{\alpha\in I}$. Let

$$p^{(k)} : T^{(k)}X \to X$$

be the projection obtained from restricting $T^{(k)}\mathbb{R}^N \to \mathbb{R}^N$ to X' and composing with the inverse of the realization homeomorphism $X \to X'$.

Define

$$N^{(k)}_\alpha := T^{(k)}\nu(S_\alpha) \cap T^{(k)}X$$

to be the tube around the stratum $T^{(k)}S_\alpha$ of $T^{(k)}X$. Let the restriction of the k-th derivative

$$d^{(k)}\pi_\alpha : T^{(k)}\nu(S_\alpha) \to T^{(k)}S_\alpha$$

to $N^{(k)}_\alpha$ be the associated tubular projection $\pi^{(k)}_\alpha := d^{(k)}\pi_\alpha$. Let $d^{(i)}\rho_\alpha : T^{(i)}\nu(S_\alpha) \to \mathbb{R}$ be the i-th derivative of the radial function ρ_α. We define the radial function associated to the stratum $T^{(k)}S_\alpha$ of $T^{(k)}X$ to be

$$\rho^{(k)}_\alpha := \rho_\alpha \circ p^{(k)} + (d\rho_\alpha)^2 + \cdots + (d^{(k)}\rho_\alpha)^2$$

restricted to $T^{(k)}_\alpha$. This defines a tube system $\mathcal{N}^{(k)} = (T^{(k)}_\alpha, \pi^{(k)}_\alpha, \rho^{(k)}_\alpha)_{\alpha\in I}$ and $(T^{(k)}X, \Sigma^{(k)}, \mathcal{N}^{(k)})$ is abstractly stratified.

Definition 2.27. *Let (E, X, p) be a stratified fiber bundle and $U \subset X$ be an open subset with the canonical abstract stratification inherited from X. We define a* **formal (weakly) controlled E-valued r-jet** *over U to be an $(r+1)$-tuple $(s_0, s_1, \cdots, s_r)$ such that $s_k : T^{(k)}U \to T^{(k)}E$ is a (weakly) controlled section of $(T^{(k)}E, T^{(k)}X, d^{(k)}p)$ over $T^{(k)}U \subset T^{(k)}X$ for all $0 \le k \le r$ and each s_{k+1}* **covers** *s_k in the sense that the following diagram commutes:*

$$\begin{array}{ccccccccc}
E & \longleftarrow & TE & \longleftarrow & T^{(2)}E & \longleftarrow & \cdots & \longleftarrow & T^{(r)}E \\
\uparrow{\scriptstyle s_0} & & \uparrow{\scriptstyle s_1} & & \uparrow{\scriptstyle s_2} & & \cdots & & \uparrow{\scriptstyle s_r} \\
U & \longleftarrow & TU & \longleftarrow & T^{(2)}U & \longleftarrow & \cdots & \longleftarrow & T^{(r)}U
\end{array}$$

An extended and modified notion of stratified jets will be given later in Definition 5.13 before we prove the $h-$principle for jet sheaves. The **sheaf of formal (weakly) controlled E-valued r-jets**, denoted as $\mathcal{J}^r_E$ ($\mathcal{J}^r_{E,w}$)), assigns to each open subset $U \subset X$ the set $\mathcal{J}^r_E(U)$ ($\mathcal{J}^r_{E,w}(U)$)) of all formal (weakly) controlled E-valued r-jets

over U. Let Γ_E ($\Gamma_{E,w}$) denote the sheaf of (weakly) controlled local sections of (E, X, p). Then there is similarly a morphism of sheaves

$$J^r : \Gamma_E \to \mathcal{J}^r_E \tag{11}$$
$$\left(J^r_w : \Gamma_{E,w} \to \mathcal{J}^r_{E,w}\right)$$

which sends a (weakly) controlled local section $s \in \Gamma_E(U)$ of (E, X, p) over an open subset $U \subset X$ to the formal (weakly) controlled E-valued r-jet

$$J^r s := (s, ds, d^{(2)}s, \cdots, d^{(r)}s) \in \mathcal{J}^r_E(U)$$
$$\left(J^r_w s := (s, ds, d^{(2)}s, \cdots, d^{(r)}s) \in \mathcal{J}^r_{E,w}(U)\right).$$

The image of this morphism is a subsheaf $\mathcal{H}^r_E$ of $\mathcal{J}^r_E$ ($\mathcal{H}^r_{E,w}$ of $\mathcal{J}^r_{E,w}$) which we shall call the **(weakly) controlled sheaf of holonomic E-valued r-jets on X**.

Let $T^{(k)}E$ be equipped with metrics $\mathrm{dist}^{(k)}_E$ respecting the topology for all $0 \leq k \leq r$. Then for any open subset $U \subset X$ we can equip $\mathcal{J}^r_E(U)$ (or $\mathcal{J}^r_{E,w}(U)$) with a metric topology: we shall call two (weakly) controlled E-valued r-jets $J = (s_0, s_1, \cdots, s_r)$ and $J' = (s'_0, s'_1, \cdots, s'_r)$ over U are **ε-close** if

$$\sup_{x \in U} \mathrm{dist}^{(k)}_E(s_k(x), s'_k(x)) < \varepsilon \text{ for all } 0 \leq k \leq r$$

3. Flexibility, Diff-invariance

Two crucial notions that come into play in Gromov's sheaf-theoretic h-principle[Gro86, Section 2.2] over manifolds are

(1) flexibility,
(2) Diff-invariance.

The purpose of this section is to extend these two notions both at the level of the base space as well as that of the nature of the sheaf. Thus, we shall

(1) replace the base manifold by a stratified space,
(2) replace the sheaf of quasi-topological spaces in [Gro86] by stratified sheaves,

and extend Gromov's notions of flexibility and Diff-invariance to this setup.

3.1. Flexibility of sheaves.

Following Gromov [Gro86, Ch. 2], we shall refer to sheaves of quasitopological spaces as *continuous sheaves*. We collect together in this subsection, some basic notions from [Gro86, Ch. 2], and facts about continuous sheaves.

Definition 3.1. [Gro86, p. 40] *Let $\alpha : A \to A'$ be a continuous map of quasitopological spaces. Consider a continuous map $\phi : P \to A$ of a compact polyhedron P into A. Let $\phi' = \alpha \circ \phi$. Let $\Phi' : P \times [0,1] \to A'$ be such that $\Phi'|P \times \{0\} = \phi'$.*

The map α is called a (Serre) fibration *if for all such polyhedra P, maps $\phi : P \to A$ and homotopies Φ' of ϕ', Φ' lifts to a map $\Phi : P \times [0,1] \to A$ such that $\Phi|P \times \{0\} = \phi$ and $\alpha \circ \Phi = \Phi'$.*

The map α is called a (Serre) microfibration *if for all such polyhedra P, maps $\phi : P \to A$ and homotopies Φ' of ϕ', there exists $0 < \varepsilon \leq 1$ (where ε may depend on P, ϕ, Φ') and a map $\Phi : P \times [0, \varepsilon] \to A$, such that $\Phi|P \times \{0\} = \phi$ and $\alpha \circ \Phi = \Phi'|P \times [0, \varepsilon]$.*

Henceforth, by fibration (resp. microfibration), we shall mean a Serre fibration (resp. microfibration) of quasitopological spaces.

Definition 3.2. *Let X be locally compact Hausdorff. A continuous sheaf $\mathcal{F}$ on X is* flexible *(resp.* microflexible*) if for all compact $K \subset K'$, $\mathcal{F}(K') \to \mathcal{F}(K)$ is a fibration (resp. microfibration).*

Example 3.3. *Let $f : Y \to X$ be surjective, and $\mathcal{F}$ be the continuous sheaf of sections associated to f, equipped with the quasitopology on mapping spaces. Then $\mathcal{F}$ is flexible.*

Theorem 3.4. [Gro86, Theorem B, p. 77] *Let $\Phi : \mathcal{F} \to \mathcal{G}$ be a morphism of flexible sheaves over a finite dimensional locally compact Hausdorff space X. Then Φ is a local weak homotopy equivalence if and only if Φ is a weak homotopy equivalence, i.e. $\Phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is a weak homotopy equivalence for all $x \in X$ if and only if $\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a weak homotopy equivalence for all $U \subset X$ open.*

3.2. **Stratified spaces and flexibility conditions.** Let (X, Σ) be a Whitney stratified space in the sense of Definition 2.3. Then, by the Whitney conditions (a), (b) of Definition 2.2 and Thom's isotopy lemma [GM88, Section 1.5] (see also Lemma 2.15) we have:

Lemma 3.5. *Let $x \in X$ and let S denote the unique stratum in which x lies. Then there exists an open neighborhood U of x and a stratum-preserving homeomorphism $\phi : U \to \mathbb{R}^i \times cA$, where*

1. *S has dimension i*
2. *A is a compact stratified space admitting a stratum-preserving homeomorphism with the link of S in X.*

We shall now define a notion of sheaves over stratified spaces (X, Σ). This is a finer notion than that of a sheaf over the underlying topological space X. It associates data to open subsets of each stratum-closure of X. To formulate this, we introduce the *stratified site* associated to (X, Σ):

Definition 3.6. *A stratified space (X, Σ) comes equipped with the canonical filtered collection of topological spaces $\{\overline{S}\}$, where*

1. *S is a stratum of (X, Σ).*
2. *$\overline{S}$ is equipped with the subspace topology inherited from X.*

The stratified site $\mathrm{Str}(X, \Sigma)$ *is the full subcategory of all open sets of X, where*

1. *Objects of $\mathrm{Str}(X, \Sigma)$ are open subsets $U \subset \overline{S}$ of some stratum-closure.*
2. *Morphisms of $\mathrm{Str}(X, \Sigma)$ are inclusions $U \hookrightarrow V$ between such subsets.*

Remark 3.7. *The term stratified site in Definition 3.6 is borrowed from algebraic geometry. For example, it can be checked that $\mathrm{Str}(X, \Sigma)$ comes equipped with a natural Grothendieck topology (namely, sieves in $\mathrm{Str}(X, \Sigma)$ are covers consisting of objects in $\mathrm{Str}(X, \Sigma)$), and forms an example of a site.*

Definition 3.8. *A* stratified continuous sheaf *$\mathcal{F}$ on X is a collection of continuous sheaves $\{\mathcal{F}_{\overline{L}}\}$, one for every stratum L of X, such that for every pair $S < L$, there is a morphism of sheaves*

$$\mathrm{res}_S^L : i_{\overline{S}}^* \mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{S}}$$

that we call the restriction map from L to S. *Thus, a stratified sheaf $\mathcal{F}$ assigns a quasitopological space $\mathcal{F}(U)$ to every object of $\mathrm{Str}(X, \Sigma)$ (Definition 3.6) in a way that the gluing axiom is satisfied, i.e. it is a quasitopological space-valued sheaf on the site $\mathrm{Str}(X, \Sigma)$.*

Open subsets of (X, Σ) are naturally stratified subsets; hence they are elements of $\mathrm{Str}(X, \Sigma)$. We shall denote by $\mathcal{F}_S$ the restriction of the sheaf $\mathcal{F}_{\overline{S}}$ to the open stratum S.

Recall (Definition 2.27 and the discussion in Section 2.4) that for any stratified fiber bundle (E, X, p), there are natural sheaves $\mathcal{J}^r_E, \mathcal{H}^r_E, \mathcal{J}^r_{E,w}, \mathcal{H}^r_{E,w}$ consisting of controlled (or weakly controlled) formal and holonomic jets.

Definition 3.9. *A stratified sheaf on (X, Σ) is* flexible *(resp.* microflexible*) if for any stratum S, $\mathcal{F}_{\overline{S}}$ is flexible (resp. microflexible).*

A stratified sheaf on (X, Σ) is stratumwise flexible *(resp.* stratumwise microflexible*) if for any stratum S, $\mathcal{F}_S$ is flexible (resp. microflexible).*

Note that the latter is a condition on the sheaves $\{\mathcal{F}_{\overline{S}}\}$ comprising the stratified sheaf $\mathcal{F}$ *after restricting each $\mathcal{F}_{\overline{S}}$ to the open stratum S.*

We recall a construction from [Gro86, p. 77]. Let $\mathcal{F}, \mathcal{G}$ be continuous sheaves on X and $q : \mathcal{F} \to \mathcal{G}$ be a morphism of continuous sheaves. Consider the continuous sheaf $\widetilde{\mathcal{F}}$ defined by assigning to every open set $U \subset X$ the set

$$\widetilde{\mathcal{F}}(U) := \{(s, \gamma) \in \mathcal{F}(U) \times \mathrm{Maps}(I, \mathcal{G}(U)) : q(s) = \gamma(0)\}.$$

Equip $\widetilde{\mathcal{F}}(U)$ with a quasitopology as follows: for any topological space W, a map $W \to \widetilde{\mathcal{F}}(U)$ is continuous if and only if the projections $W \to \mathcal{F}(U)$ and $W \to \mathrm{Maps}(I, \mathcal{G}(U))$ are continuous. There is a morphism of continuous sheaves $\widetilde{q} : \widetilde{\mathcal{F}} \to \mathcal{G}$ given by $\widetilde{q}(s, \gamma) = \gamma(1)$. Then

$$\widetilde{q} : \widetilde{\mathcal{F}}(U) \to \mathcal{G}(U)$$

is a fibration. Let $\psi \in \mathcal{G}(X)$ be a global section.

Definition 3.10. *We shall call the fiber of $\widetilde{q}$ over ψ the* homotopy fiber *of q over ψ:*

$$\mathrm{hofib}(q; \psi)(U) := \widetilde{q}^{-1}(\psi|_U) \subset \widetilde{\mathcal{F}}(U).$$

If the choice of $\psi \in \mathcal{G}(X)$ is understood, we simply denote the homotopy fiber sheaf as $\mathrm{hofib}(q)$.

Let $\mathcal{F}$ be a stratified continuous sheaf on (X, Σ). For ease of exposition, we assume the existence of and fix a global section $\psi \in \mathcal{F}(X)$.

Definition 3.11. *For $S < L$, define the* closed homotopy fiber sheaf *of $\mathcal{F}$ from L to S by*

$$\overline{\mathcal{H}}^L_S = \mathrm{hofib}(\mathrm{res}^L_S : i^*_{\overline{S}} \mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{S}}).$$

The corresponding open homotopy fiber sheaf *is defined to be $\mathcal{H}^L_S = i^*_S \overline{\mathcal{H}}^L_S$*

Note that

$$\mathcal{H}^L_S = \mathrm{hofib}(i^*_S \mathcal{F}_{\overline{L}} \to \mathcal{F}_S).$$

Definition 3.12. *A stratified continuous sheaf on (X, Σ) is* infinitesimally flexible across strata *if for any $S < L$ in Σ, $\mathcal{H}^L_S$ is flexible.*

Lemma 3.13. *Let $\mathcal{F}$ be a continuous sheaf over X and $Z \subset K \subset X$ be compact subsets.*

1. *If $\mathcal{F}(K) \to \mathcal{F}(Z)$ is a fibration, then so is $\mathrm{Maps}(I^n, \mathcal{F}(K)) \to \mathrm{Maps}(I^n, \mathcal{F}(Z))$.*
2. *If $\mathcal{F}$ is a stratumwise flexible stratified sheaf, so is $\mathrm{Maps}(I^n, \mathcal{F})$.*

(3) *If $\mathcal{F}$ is a stratified sheaf which is infinitesimally flexible across strata, so is* $\mathrm{Maps}(I^n, \mathcal{F})$.

Proof. 1) We are given that $\mathcal{F}(K) \to \mathcal{F}(Z)$ is a fibration. Let C be a CW-complex; then every map $C \times I^n \times I \to \mathcal{F}(Z)$ with a chosen initial lift $C \times I^n \times \{0\} \to \mathcal{F}(K)$ admits a lift. Thus, $\mathrm{Maps}(I^n, \mathcal{F}(K)) \to \mathrm{Maps}(I^n, \mathcal{F}(Z))$ satisfies the homotopy lifting property with respect to homotopies of maps from C. As C was arbitrary, this proves $\mathrm{Maps}(I^n, \mathcal{F}(K)) \to \mathrm{Maps}(I^n, \mathcal{F}(Z))$ satisfies the homotopy lifting property and thus is a Serre fibration.
2) This is an immediate corollary of (1).
3) Infinitesimal flexibility across strata is the statement that for all $S, L \in \Sigma_X$, $S < L$, $\mathcal{H}^L_S$ is flexible. Applying $\mathrm{Maps}(I^n, -)$ and using $\iota_S^* \mathrm{Maps}(I^n, \mathcal{F}_{\overline{L}}) = \mathrm{Maps}(I^n, \iota_S^* \mathcal{F}_{\overline{L}})$, we obtain the desired claim. □

3.3. **Diff-invariance.** We shall say that $U \subset S$ is a *relatively compact embedded open ball*, if U is an open ball and $\overline{U} \subset S$ is a compact (smoothly) embedded ball in S. Following Gromov [Gro86], we shall say that a sheaf $\mathcal{F}$ over a manifold V is Diff_c*-invariant* if it is acted on by the pseudogroup of compactly supported diffeomorphisms of V in the following sense: for every pair of relatively compact open balls $U, U' \subset V$ (i.e., $\overline{U}, \overline{U'}$ are embedded compact balls in V), and a diffeomorphism $\phi : U' \to U$, there is an isomorphism of sheaves $\psi : \phi^*(\mathcal{F}|_U) \to \mathcal{F}|_{U'}$ such that ψ is functorial in ϕ, U, U'. Recall that the pseudogroup $\mathrm{Diff}_c(V)$ of diffeomorphisms is the set of all pairs (U, f), where $U \subset V$ is an open set and f is a compactly supported diffeomorphism of M carrying U onto another open set $U' = f(U) \subset V$.

Finally, if $\mathcal{F}^1, \mathcal{F}^2$ are Diff_c-invariant sheaves over a manifold V, then a morphism of sheaves $\Phi : \mathcal{F}^1 \to \mathcal{F}^2$ is said to be Diff_c*-invariant* if it is natural with respect to the $\mathrm{Diff}_c(V)$-action, i.e. if $\psi_i : \phi^*(\mathcal{F}^i|_U) \to \mathcal{F}^i|_{U'}$, for $i = 1, 2$, denote the isomorphisms above, then the following diagram commutes:

$$\begin{array}{ccc} \phi^*(\mathcal{F}^1|_U) & \xrightarrow{\psi_1} & \mathcal{F}^1|_{U'} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ \phi^*(\mathcal{F}^2|_U) & \xrightarrow{\psi_2} & \mathcal{F}^2|_{U'} \end{array}$$

Definition 3.14. *A stratified continuous sheaf on a stratified space (X, Σ) is* StratDiff-invariant *if*

(1) *for any $S < L$ in Σ, $i_S^* \mathcal{F}_{\overline{L}}$ is $\mathrm{Diff}_c(S)-$invariant.*
(2) *for any $S \in \Sigma$, $\mathcal{F}_S$ is $\mathrm{Diff}_c(S)-$invariant.*
(3) *for any $S < L$ in Σ, res^L_S is $\mathrm{Diff}_c(S)-$invariant.*

We observe the following.

Lemma 3.15. *Let $\mathcal{F}$ be a Diff_c-invariant sheaf over a connected manifold M. Then $\mathcal{F}$ has constant stalks, i.e. for any pair of points $x, y \in M$, $\mathcal{F}_x \cong \mathcal{F}_y$.*

Proof. Let $x, y \in M$. Let $\{U_i : i \in \mathbb{N}\}$ be a family of nested open balls around x such that $\cap_i U_i = \{x\}$. There exists a homeomorphism $\phi : U_1 \to V_1$ such that V_1 is a neighborhood of y, and $\phi(x) = y$. let $V_i = \phi(U_i)$. Then $\{V_i : i \in \mathbb{N}\}$ is a family of nested open balls around y such that $\cap_i V_i = \{y\}$. By Diff_c-invariance, $\mathcal{F}|_{U_i} \cong \phi^* \mathcal{F}|_{V_i}$, and hence (by passing to limits), $\mathcal{F}_x \cong \mathcal{F}_y$. □

Let $\mathcal{F}$ be a stratified (continuous) sheaf over a stratified space (X, Σ) such that

(1) $\mathcal{F}$ is infinitesimally flexible across strata.
(2) $\mathcal{F}$ is StratDiff $-$invariant.

Recall that for any $S < L$ in Σ, $\mathrm{res}^L_S : i_S^* \mathcal{F}_{\overline{L}} \to \mathcal{F}_S$ is a morphism of sheaves, and $\mathcal{H}^L_S = \mathrm{hofib}(\mathrm{res}^L_S)$ denotes the homotopy fiber sheaf.

Lemma 3.16. *$\mathcal{H}^L_S$ is $\mathrm{Diff}_c(S)-$invariant. In particular, $\mathcal{H}^L_S$ has constant stalks over S.*

Proof. By StratDiff $-$invariance (Definition 3.14) of $\mathcal{F}$, it follows that

(1) for any $S < L$ in Σ, $i_S^* \mathcal{F}_{\overline{L}}$ is $\mathrm{Diff}_c(S)-$invariant.
(2) for any $S \in \Sigma$, $\mathcal{F}_S$ is $\mathrm{Diff}_c(S)-$invariant.
(3) for any $S < L$ in Σ, res^L_S is $\mathrm{Diff}_c(S)-$invariant.

By functoriality of the homotopy fiber construction, $\mathcal{H}^L_S$ is $\mathrm{Diff}_c(S)-$invariant. Hence, by Lemma 3.15, $\mathcal{H}^L_S$ has constant stalks. □

4. The sheaf-theoretic h-principle

4.1. **The (Gromov) diagonal normal sheaf.** Let $\mathcal{F}$ be a continuous sheaf over a locally compact countable polyhedron X (e.g. a stratified space). Define a sheaf $\mathcal{P}$ over $X \times X$ by assigning to every basic open set $U \times V \subset X \times X$, the quasitopological space

$$\mathcal{P}(U \times V) := \mathrm{Maps}(U, \mathcal{F}(V)),$$

Definition 4.1. *The (Gromov) diagonal normal sheaf $\mathcal{F}^*$ associated to $\mathcal{F}$ is defined by*

$$\mathcal{F}^* = \mathrm{diag}^* \mathcal{P},$$

where $\mathrm{diag} : X \to X \times X$ is the diagonal embedding.

When $\mathcal{F}$ is a subsheaf of the sheaf of sections of a surjective map $P : E \to X$ between topological spaces, an alternate description of the (Gromov) diagonal normal sheaf may be given in terms of (a slight relaxation of) Milnor's construction of microbundles [Mil64, Kis64].

Definition 4.2. *Let X be a topological space. The* tangent microbundle (U_X, X, p) *to X is defined to be the germ of a neighborhood U_X of $\mathrm{diag}(X) \subset X \times X$ along with the projection $p : U_X \to X$ to the first coordinate.*

Remark 4.3. *In Milnor's definition* [Mil64]*, a microbundle is required to always be locally trivial, whereas we relax this condition.*

Let $P : E \to X$ be surjective and $\Gamma(U, E)$ denote the space of sections over $U \subset X$ equipped with the compact open topology. Let $\mathcal{F}$ denote a subsheaf of the sheaf of sections $\Gamma(-, E)$ satisfying some property $\mathcal{A}$, i.e. $\mathcal{F}(U)$ consists of sections $s \in \Gamma(U, E)$ satisfying the property $\mathcal{A}$.

Then $\mathcal{P}(U \times V)$ consists of continuous maps from U to $\mathcal{F}(V) \subset \Gamma(V, E)$ (where the latter has the inherited compact open topology). The following are two equivalent descriptions:

(1) $\mathcal{P}(U \times V)$ consists of $U-$parametrized families of sections over V satisfying property $\mathcal{A}$
(2) $\mathcal{P}(U \times V)$ consists of continuous maps from $U \times V$ to E such that for each $x \in U$, it restricts on $\{x\} \times V$ to a section $\sigma_x : V \to E$ satisfying property $\mathcal{A}$

It is more convenient to think of $\mathcal{F}^*$ as the restriction of $\mathcal{P}$ to the tangent microbundle (U_X, X, p) (Definition 4.2) rather than as the restriction to $\mathrm{diag}(X)$. This is because (U_X, X, p) is defined as a germ of open neighborhoods of $\mathrm{diag}(X) \subset X \times X$, and restriction of the sheaf $\mathcal{P}$ to any representative of U_X makes sense *without passing to limits*. We proceed to describe this in some more detail.

Let $\{U_i \times U_i\}$ be a collection of basic open sets in $X \times X$. We shall say that a collection of elements $\phi_i \in \mathcal{P}(U_i \times U_i)$ are *consistent*, if for all $i \neq j$,

$$\phi_i = \phi_j \text{ on } (U_i \cap U_j) \times (U_i \cap U_j)$$

Let $W \subset \mathrm{diag}(X) \subset X \times X$ be an open subset. Then any element of $\mathcal{F}^*(W)$ is represented by a family of consistent elements $\phi_i \in \mathcal{P}(U_i \times U_i)$, where $\{U_i \times U_i\}$ covers W. Using the equivalence described in the preceding paragraphs, we may treat ϕ_i as maps $\phi_i : U_i \times U_i \to E$. The consistency condition therefore allows us to glue these to a well-defined map $\phi : (W \times W) \cap U_X \to E$. We list the properties of this map as a characterization of sections of $\mathcal{F}^*$ over W:

Remark 4.4. $\phi : (W \times W) \cap U_X \to E$ is an element of $\mathcal{F}^*(W)$ if and only if

(1) The restriction $\phi|_{\mathrm{diag}(W)}$ of ϕ to $\mathrm{diag}(W)$ is a section of E over $p(\mathrm{diag}(W)) = W \subset X$, and
(2) For any $w \in W$, the restriction of ϕ to $(\{w\} \times W) \cap U_X$ is the germ of a section of E over the open subset $p_2((\{w\} \times W) \cap U_X) \subset W \subset X$ of w where $p_2 : X \times X \to X$ defines the projection of $X \times X$ to the second coordinate.

Thus, the first coordinate in $X \times X$ defines the base space of the microtangent bundle (U_X, X, p), and the second gives germs of neighborhoods of points $x \in X$. Hence, an element of $\mathcal{F}^*(W)$ is given by a section of E over W (in the first coordinate) decorated with germs of sections $\{s_w : w \in W\}$ (in the second coordinate).

A caveat is in order. The preceding paragraph suggests that elements of $\mathcal{F}^*(X)$ correspond to maps $U_X \to E$ from the total space of the tangent microbundle (U_X, X, p) to the total space of the surjective map (E, X, P) whose sections define the sheaf $\mathcal{F}$. However, this map is **not** a fiber-preserving map; in fact, the situation is completely orthogonal. This is because the fibers of the tangent microbundle $p : U_X \to X$ are subspaces of the second factor in the square $X \times X$, which map to germs of *sections* of the surjection $P : E \to X$, and are therefore "transverse" to the fibers of P.

Definition 4.5. *Let $\mathcal{F}$ denote a continuous sheaf X. Let W be a fixed topological space. Define a new sheaf* $\mathrm{Maps}(W, \mathcal{F})$ *over X as follows. For any $U \subset X$ open, set*

$$\mathrm{Maps}(W, \mathcal{F})(U) = \mathrm{Maps}(W, \mathcal{F}(U)),$$

where $\mathrm{Maps}(W, \mathcal{F}(U))$ *is equipped with the standard quasitopology on mapping spaces.*

Lemma 4.6. *For $\mathcal{F}, X, W$ as in Definition 4.5 above, the Gromov diagonal normal sheaves satisfy*

$$(\mathrm{Maps}(W, \mathcal{F}))^* = \mathrm{Maps}(W, \mathcal{F}^*).$$

Proof. Consider the sheaf $\mathcal{P}$ on $X \times X$ given by

$$\mathcal{P}(U \times V) = \mathrm{Maps}(U, \mathrm{Maps}(W, \mathcal{F})(V)).$$

Also, let $\mathcal{P}_1$ be the sheaf on $X \times X$ given by

$$\mathcal{P}_1(U \times V) = \mathrm{Maps}(U, \mathcal{F}(V)).$$

Then,

$$\begin{aligned}\mathcal{P}(U\times V) &= \mathrm{Maps}(U,\mathrm{Maps}(W,\mathcal{F}(V))) &= \mathrm{Maps}(U\times W,\mathcal{F}(V)))\\ &= \mathrm{Maps}(W,\mathrm{Maps}(U,\mathcal{F}(V))) &= \mathrm{Maps}(W,\mathcal{P}_1(U\times V).\end{aligned}$$

Hence,

$$\begin{aligned}(\mathrm{Maps}(W,\mathcal{F}))^* = &\quad \mathrm{diag}^*\,\mathcal{P} &= \mathrm{diag}^*(\mathrm{Maps}(W,\mathcal{P}_1))\\ = &\quad \mathrm{Maps}(W,\mathrm{diag}^*\,\mathcal{P}_1) &= \mathrm{Maps}(W,\mathcal{F}^*).\end{aligned}$$

□

There exists a sheaf over $W\times X$ closely related to the sheaf $\mathrm{Maps}(W,\mathcal{F})$ over X (Definition 4.5). This is defined below.

Definition 4.7. *Let $\mathcal{G}$ denote a sheaf of topological spaces over X. Let W be a fixed topological space. Define a new sheaf of $W-$parametrized sections $\mathcal{F}=\mathrm{Maps}^p(W,\mathcal{G})$ over $W\times X$ as follows. For any $U\subset X$ and $V\subset W$ open, set*

$$\mathcal{F}(V\times U)=\mathrm{Maps}^p(W,\mathcal{F})(V\times U)=\mathrm{Maps}(V,\mathcal{G}(U)),$$

where $\mathrm{Maps}(V,\mathcal{G}(U))$ is equipped with the compact open topology.

Example 4.8. A natural example of a sheaf of $W-$parametrized sections may be given by the following. Let $P:Y\to X$ be a continuous surjective map and let $\mathcal{G}$ denote the sheaf of continuous sections of P. Let W be a fixed topological space. Define a continuous surjective map $P_W: W\times Y\to W\times X$ such that $P_W(w,y)=(w,P(y))$. Then the sheaf of continuous sections of P_W is given precisely by $\mathcal{F}=\mathrm{Maps}^p(W,\mathcal{G})$.

Lemma 4.9. *For $\mathcal{F},\mathcal{G},X,W$ as in Definition 4.7 above, the Gromov diagonal normal sheaves satisfy the following for open $V\subset W$ and $U\subset X$:*

$$\mathcal{F}^*(V\times U)=\mathrm{Maps}(V,\mathcal{G}^*(U)).$$

Proof. As in Definition 4.1, the Gromov diagonal normal sheaf is constructed by first constructing a sheaf $\mathcal{P}$ on $(W\times X)\times(W\times X)$ as follows.

$$\begin{aligned}\mathcal{P}((V\times U)\times(V\times U)) = &\quad \mathrm{Maps}((V\times U),\mathcal{F}(V\times U))\\ = &\quad \mathrm{Maps}((V\times U),\mathrm{Maps}(V,\mathcal{G}(U)))\\ = &\quad \mathrm{Maps}((V\times V),\mathrm{Maps}(U,\mathcal{G}(U)))\end{aligned}$$

Restricting $\mathcal{P}$ to the diagonal, we have the following.

$$\begin{aligned}\mathcal{F}^*(V\times U) = &\quad \mathrm{diag}^*\,\mathcal{P}((V\times U)\times(V\times U))\\ = &\quad \varinjlim_{\{\mathcal{O}:\,(V\times U)\times(V\times U)\supset\mathcal{O}\supset\mathrm{diag}(V\times U)\}}\mathcal{P}((V\times U)\times(V\times U))\\ = &\quad \varinjlim_{\{\mathcal{O}:\,(V\times U)\times(V\times U)\supset\mathcal{O}\supset\mathrm{diag}(V\times U)\}}\mathrm{Maps}((V\times V),\mathrm{Maps}(U,\mathcal{G}(U)))\\ = &\quad \mathrm{Maps}(V,\mathcal{G}^*(U))\end{aligned}$$

This completes the proof. □

There is a tautological inclusion

$$\Delta:\mathcal{F}\xrightarrow{\iota}\mathcal{F}^*$$

sending $s\in\mathcal{F}(U)$ to $(s,\{s_x:x\in U\})$, where s_x denotes the germ of the section s at $x\in U$.

Definition 4.10. [Gro86, p. 76] *A (continuous) sheaf $\mathcal{F}$ satisfies the sheaf theoretic h-principle, if every section $\phi \in \mathcal{F}^*(U)$ can be homotoped to $\mathcal{F}(U) \subset \mathcal{F}^*(U)$ for all open subsets $U \subset V$. Further, $\mathcal{F}$ satisfies the parametric sheaf theoretic h-principle if the morphism $\Delta_U : \mathcal{F}(U) \to \mathcal{F}^*(U)$ is a weak homotopy equivalence for all open $U \subset X$.*

Proposition 4.11. [Gro86, p. 76] *Let $\mathcal{F}$ be a continuous sheaf over a locally compact finite dimensional Hausdorff space X. Then $\mathcal{F}^*$ is flexible.*

Theorem 4.12. [Gro86, p.76] *Let $\mathcal{F}$ be a continuous sheaf over a locally compact countable polyhedron X (e.g. a manifold or a stratified space). If $\mathcal{F}$ is flexible, it satisfies the parametric h-principle.*

Remark 4.13. For sheaves, the notion of flexibility is strictly stronger than that of an $h-$principle. Suppose X is locally contractible. By definition of the stalks $\mathcal{F}_x$ and $\mathcal{F}^*_x$ of a continuous sheaf $\mathcal{F}$ and its diagonal normal form $\mathcal{F}^*$, respectively, we see that the tautological inclusion $\Delta_x : \mathcal{F}_x \to \mathcal{F}^*_x$ is a weak homotopy equivalence at the level of stalks. That is, the germ of a formal section at x is homotopic to the germ of a holonomic section at x. Therefore, given a formal section in an open neighborhood U_x of x, we may homotope it to a holonomic section, possibly in a small open neighborhood $U'_x \subset U_x$. Flexibility allows us to glue these local holonomic sections together to obtain a (global) holonomic section over a large open set. Thus, flexibility may be thought of as an analog of a Mayer-Vietoris principle used to glue homotopy equivalences (see for instance Theorem 4.31 below).

The existence of the $h-$principle is invariant under homotopy equivalence of sheaves (essentially by definition), i.e. if $\mathcal{F}$ satisfies the $h-$principle and $\mathcal{G}$ is homotopy equivalent to $\mathcal{F}$, then $\mathcal{G}$ satisfies the $h-$principle. The same is not true for flexibility. This is the raison d'etre behind the existence of Section 2.2.7 of [Gro86].

4.2. **Parametric h-principle for stratified continuous sheaves.** We now extend the notion of a sheaf-theoretic h-principle (Definition 4.10) to stratified sheaves. The essential difference between a stratified continuous sheaf and a continuous sheaf over X is that a stratified sheaf $\mathcal{F}$ assigns a quasitopological space $\mathcal{F}(U)$ to an open subset of $U \subset \overline{L}$ for *every* stratum L of X, whereas an ordinary sheaf does so only for open subsets of X. Hence, for every pair $S < L$, and for every $x \in S$, there are two stalks:

(1) $(\mathcal{F}_S)_x$, which we shall refer to as the *intrinsic stalk*. More generally, for any $U \subset S$, $(\mathcal{F}_S)|_U$ will be referred to as the *intrinsic sheaf* over U.
(2) $(i_S^*\mathcal{F}_{\overline{L}})_x$, which we shall refer to as the *extrinsic stalk*. More generally, for any $U \subset S$, $(i_S^*\mathcal{F}_{\overline{L}})|_U$ will be referred to as the *extrinsic sheaf* over U.

Definition 4.14. *For a stratified sheaf $\mathcal{F} = \{\mathcal{F}_{\overline{S}}\}$, over a stratified space X, the Gromov diagonal normal stratified sheaf is given by $\mathcal{F}^* = \{\mathcal{F}^*_{\overline{S}}\}$.*

We check below that the restriction morphisms of $\mathcal{F}^*$ are the expected ones, and that there is a canonical map from $\mathcal{F}$ to $\mathcal{F}^*$.

Lemma 4.15. *$\mathcal{F}^*$ is a stratified sheaf, and $\Delta : \mathcal{F} \to \mathcal{F}^*$ is a morphism of stratified sheaves.*

Proof. The intrinsic and extrinsic sheaves give two different diagonal normal sheaves (Definition 4.1) $\mathcal{F}^*_S$ and $(i_S^*\mathcal{F}_{\overline{L}})^*$ respectively. We note also that for open subsets U of $\overline{L}$ (for any stratum L), the morphism $\Delta : \mathcal{F}|U \to \mathcal{F}^*|U$ is a morphism of sheaves.

Hence, it suffices to show that for $S < L$, and $V \subset S$, $\mathrm{res}_S^{L^*} : i^*(\mathcal{F}_{\overline{L}})^* \to \mathcal{F}_S^*$ is a morphism of (continuous) sheaves (for the purposes of this proof we use i in place of i_S as no other strata are involved). It will immediately follow from the definition of $\mathcal{F}^*$, that the following diagram commutes for $S < L$:

$$\begin{array}{ccc} i^*\mathcal{F}_L|V & \xrightarrow{\Delta} & (i^*\mathcal{F}_L)^*|V \\ {\scriptstyle \mathrm{res}_S^L}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_S^{L*}} \\ \mathcal{F}_S|V & \xrightarrow{\Delta} & \mathcal{F}_S^*|V \end{array}$$

In other words, we need to show that the restriction map res_S^L induces a natural restriction map $\mathrm{res}_S^{L^*}$ between the Gromov diagonal normal extrinsic sheaf $(i^*\mathcal{F}_L)^*$ and the Gromov diagonal normal intrinsic sheaf $\mathcal{F}_S^*$. This is an exercise in unwinding definitions.

On $X \times X$ define $\mathcal{P}_L(U \times V) = \mathrm{Maps}(U, \mathcal{F}(V))$ where U, V are any pair of open subsets in some stratum-closure $\overline{L}$. The collection

$$\{\mathcal{P}_L : L \,\text{a stratum of X}\}$$

defines a stratified sheaf $\mathcal{P}$ on $X \times X$. Restricting $\mathcal{P}$ to $\mathrm{diag}(X)$ we get the Gromov normal sheaf $\mathcal{F}^*$. Note that $\overline{L} \times \overline{L} \subset X \times X$ comes with the product stratification, and $\mathrm{diag}(S) \subset \mathrm{diag}(\overline{L})$ gives the diagonally embedded stratum S in the diagonally embedded $\overline{L}$.

It remains to show that the diagonal restriction of $\mathcal{P}$ is a *stratified sheaf*. We have four strata $S \times S, S \times L, L \times S, L \times L \subset \overline{L} \times \overline{L}$ in $\overline{L} \times \overline{L}$ that need be considered in defining the restriction of $\mathcal{P}$ to any subset that inherits its stratification from the product stratification of $S < L$.

Observation 4.16. *The definition of $\mathcal{F}^*$ only considers the strata $S \times S, L \times L$. The strata $S \times L, L \times S$ are irrelevant.*

Proof of Observation 4.16. We explain this observation in some detail. For $U, V \subset S$, we shall denote open neighborhoods in $\overline{L}$ by U_L, V_L respectively. For $K \subset S$,

$$\mathcal{F}_S^*(K) = \varinjlim_{U \supset K, V \supset K} \mathcal{P}_S(U \times V) = \varinjlim_{U \supset K, V \supset K} \mathrm{Maps}(U, \mathcal{F}(V)),$$

whereas

$$\mathcal{F}_L^*(K) = \varinjlim_{U_L \supset K, V_L \supset K} \mathcal{P}_L(U_L \times V_L) = \varinjlim_{U_L \supset K, V_L \supset K} \mathrm{Maps}(U_L, \mathcal{F}(V_L)).$$

Thus, while there are, in general, four limits to be considered (corresponding to the four strata $S \times S, S \times L, L \times S, L \times L \subset \overline{L} \times \overline{L}$), the definition of $\mathcal{F}^*$ only needs $S \times S, L \times L$. □

Finally, we can assume without loss of generality that $U = U_L \cap S$ and $V = V_L \cap S$. This gives restriction maps from $\mathcal{P}_L(U_L \times V_L) \to \mathcal{P}_S(U \times V)$. Passing to direct limits furnishes $\mathrm{res}_S^{L^*}$ concluding the proof. □

We now have the following analog of Definition 4.10:

Definition 4.17. *A (continuous) stratified sheaf $\mathcal{F}$ satisfies the stratified sheaf theoretic h-principle, if every stratified section $\phi \in \mathcal{F}^*(U)$ can be homotoped through stratified sections to $\mathcal{F}(U) \subset \mathcal{F}^*(U)$ for all open subsets $U \subset V$.*

Further, $\mathcal{F}$ satisfies the parametric stratified sheaf theoretic h-principle if the morphism $\Delta_U : \mathcal{F}(U) \to \mathcal{F}^*(U)$ of stratified sheaves (furnished by Lemma 4.15) is a weak homotopy equivalence for all open $U \subset X$ equipped with the inherited stratification, i.e. the morphism $\Delta_U : \mathcal{F}_{\overline{L}}(U \cap \overline{L}) \to \mathcal{F}^*_{\overline{L}}(U \cap \overline{L})$ given by Lemma 4.15 is a weak homotopy equivalence for every stratum L.

The following is an analog of Proposition 4.11 for stratified sheaves:

Proposition 4.18. *Let $\mathcal{F}$ be a stratified continuous sheaf over a stratified space X. Then $\mathcal{F}^*$ is flexible.*

Proof. Flexibility of $\mathcal{F}_{\overline{L}}$ for every stratum L follows from Proposition 4.11 and naturality of restriction maps from Lemma 4.15. □

We are now in a position to state the the stratified analog of Theorem 4.12.

Theorem 4.19. *Let $\mathcal{F}$ be a stratified (continuous) sheaf over a stratified space X. If $\mathcal{F}$ is flexible, it satisfies the parametric sheaf-theoretic stratified h-principle.*

Proof. By Lemma 4.15, Δ is a morphism of stratified sheaves. The weak homotopy equivalence property for $\Delta : \mathcal{F}_{\overline{L}} \to \mathcal{F}^*_{\overline{L}}$ for every stratum L follows from Theorem 4.12. □

4.3. **Topological properties.** This subsection is rather general in flavor and sets up some basic homotopy theoretic properties of continuous sheaves that will be useful later. All topological spaces in this subsection are locally compact σ-compact finite dimensional locally contractible spaces.

Definition 4.20. *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be continuous sheaves on a topological space X. We say that*

$$\mathcal{F}_1 \xrightarrow{p} \mathcal{F}_2 \xrightarrow{q} \mathcal{F}_3$$

is a homotopy fiber sequence *if*

1. *there exists some $\psi \in \mathcal{F}_3(X)$ such that $q_U \circ p_U : \mathcal{F}_1(U) \to \mathcal{F}_3(U)$ is the constant map to ψ_U, for all $U \subset X$, and*
2. *$\mathcal{F}_1 \to \mathrm{hofib}(q; \psi)$ is a weak homotopy equivalence.*

The following was observed by Gromov [Gro86, p.77] (see the paragraph preceding Theorem B' there).

Remark 4.21. *Let $\mathcal{F}, \mathcal{G}$ be continuous sheaves on X and $q : \mathcal{F} \to \mathcal{G}$ be a morphism of continuous sheaves. If $\mathcal{F}, \mathcal{G}$ are flexible, then for any $\psi \in \mathcal{G}(X)$, $\mathrm{hofib}(q; \psi)$ is flexible.*

Lemma 4.22. *Let*

$$\mathcal{F}_1 \xrightarrow{p} \mathcal{F}_2 \xrightarrow{q} \mathcal{F}_3$$

be a homotopy fiber sequence as in Definition 4.20. If $\mathcal{F}_1, \mathcal{F}_3$ satisfy the parametric h-principle, then so does $\mathcal{F}_2$.

Proof. From the homotopy fiber sequence, we obtain a sequence of morphisms $\mathcal{P}_1 \to \mathcal{P}_2 \to \mathcal{P}_3$ of continuous sheaves over $X \times X$ (see Definition 4.1 and the preceding discussion for notation). Restricting to $\mathrm{diag}(X) \subset X \times X$, we obtain a sequence of morphisms $\mathcal{F}_1^* \to \mathcal{F}_2^* \to \mathcal{F}_3^*$ of sheaves over X. By functoriality of the diagonal normal construction (see for instance Lemma 4.15), we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_1^* & \longrightarrow & \mathcal{F}_2^* & \longrightarrow & \mathcal{F}_3^* \end{array}$$

As $\mathcal{F}_1, \mathcal{F}_3$ satisfies the parametric h-principle, the first and third vertical arrows are weak homotopy equivalences. We may evaluate the diagram of sheaves on any open set $U \subset X$, and use naturality of homotopy long exact sequences corresponding to the rows to conclude, by an application of the 5-lemma, that $\mathcal{F}_2(U) \to \mathcal{F}_2^*(U)$ is a weak homotopy equivalence. Thus, $\mathcal{F}_2 \to \mathcal{F}_2^*$ is a weak homotopy equivalence of continuous sheaves. This demonstrates the parametric h-principle for $\mathcal{F}_2$. □

Remark 4.23. *The proof of Lemma 4.22 goes through mutatis mutandis to show that if any two of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ satisfy the parametric h-principle, then so does the third.*

Convention 4.24. *Henceforth, we adopt Gromov's convention* [Gro86, Section 1.4.1] *of referring to an arbitrarily small but non-specified neighborhood of a set $K \subset X$ by $\operatorname{Op}_X K$, or simply $\operatorname{Op} K$ if there is no scope for confusion. Thus, $\operatorname{Op} K$ refers to a small neighborhood of K which may become even smaller in the course of the argument* [Gro86, p. 35] *(see the table on* [Gro86, p. 36] *for further details about this convention/notation).*

Lemma 4.25. *Let $\mathcal{F}$ be a continuous sheaf on a topological space X, and $Z \subset X$ be a closed subspace. Then the diagonal normal construction commutes with restriction, i.e. there is a weak homotopy equivalence of continuous sheaves $\iota_Z^*(\mathcal{F}^*) \to (\iota_Z^*\mathcal{F})^*$.*

Proof. Recall that the diagonal normal construction applied to any sheaf yields a flexible sheaf (Proposition 4.11). Therefore $\mathcal{G} := \mathcal{F}^*$ is a flexible sheaf on X. For $K \subset Z$ compact, we have the following:

$$(\iota_Z^*\mathcal{G})(K) = \varinjlim_{\substack{V \supset K \\ V \subset Z \text{ open}}} \iota_Z^*\mathcal{G}(V) = \varinjlim_{\substack{V \supset K \\ V \subset Z \text{ open}}} \varinjlim_{\substack{U \supset V \\ U \subset X \text{ open}}} \mathcal{G}(U) = \varinjlim_{\substack{U \supset K \\ U \subset X \text{ open}}} \mathcal{G}(U) = \mathcal{G}(K)$$

Thus, flexibility of $\mathcal{G}$ implies flexibility of $\iota_Z^*\mathcal{G} = \iota_Z^*(\mathcal{F}^*)$. Moreover, $(\iota_Z^*\mathcal{F})^*$ is flexible by flexibility of the diagonal normal construction as mentioned above. Consider the sheaf morphism

$$(-)|_Z : \iota_Z^*(\mathcal{F}^*) \to (\iota_Z^*\mathcal{F})^*$$

defined on the stalk over $z \in Z$ by sending a germ of a mapping $\psi : \operatorname{Op}_X(z) \to \mathcal{F}_z$ to its restriction $\psi|\operatorname{Op}_Z(z) : \operatorname{Op}_Z(z) \to \mathcal{F}_z$. We check that $(-)|_Z$ is a sheaf morphism. Indeed, given any open set $U \subset Z$ consider an open cover $\{V_i\}$ of U in X, and a collection $\{\phi_i : V_i \to \mathcal{F}(V_i)\}$ which is *consistent*, i.e.,

$$\operatorname{res}_{V_i \cap V_j, V_i} \circ \phi_i|_{V_i \cap V_j} \equiv \operatorname{res}_{V_i \cap V_j, V_j} \circ \phi_j|_{V_i \cap V_j}.$$

Thus $\{\phi_i : V_i \to \mathcal{F}(V_i)\}$ represents an element $\phi \in \iota_Z^*(\mathcal{F}^*)(U)$. The restrictions $\{\phi_i : V_i \cap Z \to \mathcal{F}(V_i)\}$ are also consistent, simply by restricting the above equality to Z. Therefore $\{\phi_i : V_i \cap Z \to \mathcal{F}(V_i)\}$ represents an element $\phi|_Z \in (\iota_Z^*\mathcal{F})^*(U)$. Observe that $(-)|_Z$ is a *stalkwise* weak homotopy equivalence as X, Z are locally contractible. As both the domain and target sheaves are flexible, we conclude that $(-)|_Z$ is a weak homotopy equivalence by appealing to a theorem of Gromov [Gro86, Theorem B, p. 77] which says that local weak homotopy equivalence implies weak homotopy equivalence for flexible sheaves. □

Remark 4.26. Suppose $Z \subset X$ is a neighborhood deformation retract, and let $\pi : N_Z \to Z$ be a choice of such a retract. Then we can write down an explicit homotopy-inverse

$$(-) \circ \pi : (\iota_Z^* \mathcal{F})^* \to \iota_Z^*(\mathcal{F}^*)$$

defined on the stalk over $z \in Z$ by sending a germ of a mapping $\psi : \mathrm{Op}_Z(z) \to \mathcal{F}_z$ to $\psi \circ \pi : \mathrm{Op}_X(z) \to \mathcal{F}_z$. Let $\phi \in (\iota_Z^* \mathcal{F})^*(U)$ be a section represented by a consistent family $\{\phi_i : V_i \cap Z \to \mathcal{F}(V_i)\}$ for a π-saturated open cover $\{V_i\}$ (i.e., $V_i = \pi^{-1}(V_i \cap Z)$) of U in N_Z, satisfying the consistency relations:

$$\mathrm{res}_{V_i \cap V_j, V_i} \circ \phi_i|_{V_i \cap V_j \cap Z} \equiv \mathrm{res}_{V_i \cap V_j, V_j} \circ \phi_j|_{V_i \cap V_j \cap Z}$$

By taking a sufficiently fine open cover, we may assume $V_i \subset N_Z$. By composing with π on both sides we obtain a consistent family $\{\phi_i \circ \pi : V_i \to \mathcal{F}(V_i)\}$ representing $\phi \circ \pi \in (\iota_Z^*)(\mathcal{F}^*)(U)$, proving that it is a well-defined sheaf homomorphism. Gromov's theorem [Gro86, Theorem B, p. 77] once again demonstrates that it is a weak homotopy equivalence.

Definition 4.27. *Let (A, B) be a pair of topological spaces where $B \subset A$ is closed. Let $\mathcal{F}$ be a continuous sheaf on A. We define the space of sections on a* deleted germinal neighborhood of B in A *as*

$$\mathcal{F}(\mathrm{Op}(B) \setminus B) := \varinjlim_{U \supset B} \mathcal{F}(U \setminus B)$$

There is a restriction map $\mathcal{F}(U) \to \mathcal{F}(U \setminus B)$ for every open neighborhood U of $B \subset A$ which is compatible with the associated directed system indexed by the poset of open neighborhoods $\{U \subset B : A \subset U\}$. Hence we get a restriction map $\mathcal{F}(B) \to \mathcal{F}(\mathrm{Op}(B) \setminus B)$ by applying direct limits $\varinjlim_U$ to both sides. Moreover, we have a restriction map $\mathcal{F}(A \setminus B) \to \mathcal{F}(\mathrm{Op}(B) \setminus B)$ by restricting a section on $A \setminus B$ to a deleted germinal neighborhood of B in A.

Lemma 4.28. *Let (A, B) be a pair of topological spaces where $B \subset A$ is closed. Let $\mathcal{F}$ be a continuous sheaf on A. Then the following is a fiber square of quasitopological spaces:*

$$\begin{array}{ccc} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(B) \\ \downarrow & & \downarrow \\ \mathcal{F}(A \setminus B) & \longrightarrow & \mathcal{F}(\mathrm{Op}(B) \setminus B) \end{array}$$

Proof. Suppose $\psi_1 \in \mathcal{F}(B)$ and $\psi_2 \in \mathcal{F}(A \setminus B)$ such that $\psi_1|(\mathrm{Op}(B) \setminus B) = \psi_2|(\mathrm{Op}(B) \setminus B)$. Pick a representative $\widetilde{\psi}_1 \in \mathcal{F}(U)$ for some $U \supset B$ open neighborhood. Then we must have $\widetilde{\psi}_1|(V \setminus B) = \psi_2|(V \setminus B)$ for some deleted open neighborhood $V \setminus B \subset A$. Next, we know that the following is a fiber square by the gluing axiom:

$$\begin{array}{ccc} \mathcal{F}(A) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(A \setminus B) & \longrightarrow & \mathcal{F}(V \setminus B) \end{array}$$

We may then glue $\widetilde{\psi}_1|V$ and ψ_2 to obtain a $\psi \in \mathcal{F}(A)$. The element ψ is independent of the choice of $\widetilde{\psi}_1$. We therefore obtain a well-defined map

$$\Psi : \mathcal{F}(A \setminus B) \times_{\mathcal{F}(\mathrm{Op}(B) \setminus B)} \mathcal{F}(B) \to \mathcal{F}(A),$$

given by $\Psi(\psi_1, \psi_2) = \psi$. For any topological space X, consider a continuous map $f : X \to \mathcal{F}(A \setminus B) \times_{\mathcal{F}(\mathrm{Op}(B)\setminus B)} \mathcal{F}(B)$ with respect to the quasitopology on the codomain. Then f is equivalent to a pair of continuous maps $f_1 : X \to \mathcal{F}(A \setminus B)$ and $f_2 : X \to \mathcal{F}(B)$ which agree when composed with the restriction to $\mathcal{F}(\mathrm{Op}(B) \setminus B)$, by definition of quasitopology of fiber products. By definition of quasitopology on limits, there exists an open neighborhood U and a deleted open neighborhood $V \setminus B$ of B contained in U, such that f_2 factors through a continuous map $\widetilde{f_2} : X \to \mathcal{F}(U)$ and $f_1, \widetilde{f_2}$ agree when restricted to $\mathcal{F}(V \setminus B)$. Therefore, we may paste $\widetilde{f_2}|V$ and f_1 to a continuous map $g : X \to \mathcal{F}(A)$. We see that $g = f \circ \Psi$, therefore Ψ preserves the quasitopologies on the domain and codomain, i.e., Ψ is continuous.

Finally, Ψ is inverse to the natural continuous map going in the opposite direction obtained from the universal property of fiber products. Thus, Ψ is a homeomorphism of quasitopological spaces. □

Lemma 4.29. *Let $\mathcal{F}$ be a flexible sheaf on a topological space on X. Let $A, K \subset X$ be a pair of subsets such that $K, A \cap K$ are both compact. Then $\mathcal{F}(A \cup K) \to \mathcal{F}(A)$ is a fibration.*

Proof. The following is a fiber square of quasitopological space

$$\begin{array}{ccc} \mathcal{F}(A \cup K) & \xrightarrow{\mathrm{res}_{K, A\cup K}} & \mathcal{F}(K) \\ {\scriptstyle \mathrm{res}_{A, A\cup K}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{A\cap K, K}} \\ \mathcal{F}(A) & \xrightarrow{\mathrm{res}_{A\cap K, A}} & \mathcal{F}(A \cap K) \end{array}$$

Suppose $\psi : W \times I \to \mathcal{F}(A)$ is a homotopy with an initial lift $\widetilde{\psi_0} : W \times 0 \to \mathcal{F}(A \cap K)$. As $A \cap K, K$ are compact, $\mathrm{res}_{A\cap K, K} : \mathcal{F}(K) \to \mathcal{F}(A \cap K)$ is a fibration. Therefore, we may lift $\mathrm{res}_{A\cap K, A} \circ \psi : W \times I \to \mathcal{F}(A \cap K)$ with initial condition $\mathrm{res}_{K, A\cup K} \circ \widetilde{\psi_0} : W \times 0 \to \mathcal{F}(K)$ to a homotopy $\varphi : W \times I \to \mathcal{F}(K)$. Finally, using the fact that the diagram is a fiber square, ψ, φ provide a lift $\widetilde{\psi} : W \times I \to \mathcal{F}(A \cup K)$ of ψ, finishing the proof. □

Lemma 4.30. *Let $\{X_n\}$ be an inverse system and $\{Y_n\}$ be a directed system of quasitopological spaces. Let Z, W be quasitopological spaces. Let $\{f_n : X_n \to Z\}$ and $\{g_n : W \to Y_n\}$ be a collection of maps compatible with the systems $\{X_n\}$ and $\{Y_n\}$ respectively. Let $X = \varprojlim X_n$, $Y = \varinjlim Y_n$, and $f : X \to Z$, $g : W \to Y$ be the canonical maps from, and to, the respective limits.*

1. *If f_n are fibrations and the structure maps in $\{X_n\}$ are also fibrations, then f is a fibration.*
2. *If g_n are fibrations, then g is a fibration.*

Proof. Let Q be an auxiliary topological space. Let $\psi : Q \times I \to Z$ be a homotopy with an initial lift $\widetilde{\psi_0} : Q \times \{0\} \to X$ of $\psi_0 := \psi|Q \times \{0\}$. Choose a lift of ψ to a homotopy $Q \times I \to X_1$ along the fibration $f_1 : X_1 \to Z$, given initial condition $\pi_1 \circ \widetilde{\psi_0} : Q \times \{0\} \to X_1$. Then, since the structure maps of the inverse system are fibrations, we may lift $Q \times I \to X_1$ to a homotopy $Q \times I \to X_n$ using as initial condition the maps $\pi_n \circ \widetilde{\psi_0}$ for all $n \geq 2$. This gives a collection of homotopies $\{Q \times I \to X_n\}$ compatible with the structure maps of the inverse system. By the universal property of inverse limits, this provides a homotopy $\widetilde{\psi} : Q \times I \to X$, such that $\widetilde{\psi}$ is a lift of ψ with initial condition $\widetilde{\psi_0}$ as desired. This proves (1).

Let $\psi : Q \times I \to Y$ be a homotopy with an initial lift $\widetilde{\psi}_0 : Q \times \{0\} \to W$ of $\psi_0 := \psi | Q \times \{0\}$. By definition of the quasitopology of the direct limit, ψ factors through a homotopy $Q \times I \to Y_n$. Since $g_n : W \to Y_n$ is a fibration, we may lift ψ to $Q \times I \to W$, using $\widetilde{\psi}_0 : Q \times \{0\} \to W$ as the initial condition. This is a lift of ψ with initial condition $\widetilde{\psi}_0$, as desired. This proves (2). □

We shall need the following theorem to 'coglue' weak homotopy equivalences. We refer the reader to [BH70] by Brown and Heath, and the notes [Fra13, Fra11] for a proof.

Theorem 4.31. *Consider a commutative diagram of maps of quasitopological spaces as in Figure 1, where*

(1) *the front and back squares are* (strict) *pullback diagrams (equivalently, Q and P are fiber-products),*
(2) *p, q are fibrations*

If the diagonal arrows, labeled ϕ_1, ϕ_2, ϕ are weak homotopy equivalences, so is Φ.

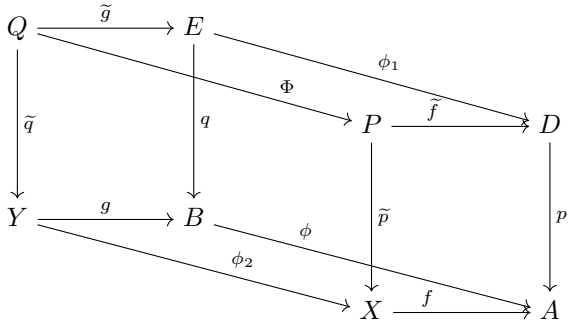


FIGURE 1. Homotopy Co-gluing

We note here for later use, a fact about homotopy fibers of fiber-products:

Lemma 4.32. *Let $f : X \to Z$, and $g : Y \to Z$ be continuous maps, furnishing the following pullback diagram:*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{F} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

Then, the homotopy fibers of $X \times_Z Y \to Y$ and $f : X \to Z$ are homotopy equivalent.

Proof. There are two homotopy fiber bundles that can be constructed over Y as follows:

(1) Let $\mathcal{P}(X \times_Z Y, F, Y) \to Y$ denote the path space fibration construction applied to $F : X \times_Z Y \to Y$, and let $P_1 : E_1 \to Y$ denote the resulting fibration. Then $\mathrm{hofib}(P_2)$ is homotopy equivalent to $\mathrm{hofib}(F)$.
(2) Let $\mathcal{P}(X, f, Z) \to Z$ denote the path space fibration construction applied to $f : X \to Z$, and let $P_2 : E_2 \to Y$ denote the pullback fibration (under g) of $\mathcal{P}(X, f, Z)$. Then $\mathrm{hofib}(P_2)$ is homotopy equivalent to $\mathrm{hofib}(f)$.

Then there exists a homotopy equivalence $E_1 \to X \times_Z Y$ covering the identity map over Y. The same holds for E_2. Hence, there exists a homotopy equivalence $\phi : E_1 \to E_2$ of total spaces covering the identity map over Y. Since $P_1 : E_1 \to Y$, and $P_2 : E_2 \to Y$ are fibrations, hofib(P_1) is homotopy equivalent to hofib(P_2). The Lemma follows. □

4.4. **Sheaf-theoretic h-principle for stratified spaces.** Let $(X, \Sigma, \mathcal{N})$ be an abstractly stratified set, and $\mathcal{F}$ be a stratified continuous sheaf on (X, Σ). For every stratum $S \in \Sigma$, we denote the associated sheaf on the closure $\overline{S} \subset X$ by $\mathcal{F}_{\overline{S}}$ so that $\mathcal{F}_S := \iota_S^* \mathcal{F}_{\overline{S}}$. For every pair of strata $S, L \in \Sigma$, $S < L$, we also have the restriction map from the sheaf on $\overline{L}$ to that on $\overline{S} \subset \overline{L}$ given by $\mathrm{res}^L_S : \iota_{\overline{S}}^* \mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{S}}$. Recall also the restriction map $\mathrm{res}^L_S : \iota_S^* \mathcal{F}_{\overline{L}} \to \mathcal{F}_S$.

4.4.1. *Limits and preliminary gluing.*

Lemma 4.33. *Let $U \subset \overline{L}$ be an open subset and $U' := U \cap S$. Let $\mathcal{G} = \mathcal{F}_{\overline{L}}$, or $\mathcal{F}^*_{\overline{L}}$. Suppose that $\mathcal{G}$ is flexible. Then $\mathcal{G}(U \setminus U') \to \mathcal{G}(\mathrm{Op}_U(U') \setminus U')$ is a fibration.*

Proof. Let $\{K_n\}$ be an ascending sequence of compact subsets of $U \setminus U'$ exhausting $U \setminus U'$. Let $\{C_n\}$ be closures (in U) of a descending sequence of open neighborhoods of U' in U such that $\cap_n C_n = U'$, i.e.

$$U' \subset \cdots \subset C_3 \subset C_2 \subset C_1 \subset U$$
$$K_1 \subset K_2 \subset K_3 \subset \cdots \subset U \setminus U'$$

Consider restriction maps $r_{m,n} : \mathcal{G}(K_m \cup (C_n \setminus U')) \to \mathcal{G}(C_n \setminus U')$. Since both K_m and $K_m \cap (C_n \setminus U') = K_m \cap C_n$ are compact, flexibility of $\mathcal{G}|L$ on L then shows that for all m, n, $r_{m,n}$ is a fibration by Lemma 4.29. We observe now that:

Claim 4.34. *$\{\mathcal{G}(K_m \cup (C_n \setminus U'))\}_m$ is an inverse system with structure maps given by fibrations.*

Proof of Claim 4.34: Observe that for all $m \geq 1$, $K_{m+1} \cup (C_n \setminus U') = (K_m \cup C_n \setminus U') \cup K_{m+1}$. Moreover, $(K_m \cup C_n \setminus U') \cap K_{m+1} = K_m \cup (K_{m+1} \cap C_n)$ and K_{m+1} are both compact. Therefore, Lemma 4.29 applies, and we obtain

$$\mathcal{G}(K_{m+1} \cup (C_n \setminus U')) \to \mathcal{G}(K_m \cup (C_n \setminus U'))$$

is a fibration, for all $m \geq 1$. □

Claim 4.35. $\varprojlim_m \mathcal{G}(K_m \cup (C_n \setminus U')) \cong \mathcal{G}(U \setminus U')$.

Proof of Claim 4.35: This is true in complete generality. Let V be an arbitrary open set (here $V = (U \setminus U')$). Let $\{C_n\}$ (resp. $\{U_n\}$) be a sequence of closed (resp. open) subsets of U such that

(1) $C_n \subset U_n \subset C_{n+1}$,
(2) $\cup_n C_n = U = \cup_n U_n$.

There exist restriction maps $\mathcal{G}(V) \to \mathcal{G}(C_n)$ which are compatible with the inverse system $\{\mathcal{G}(C_n)\}$. Hence by the universal property, there exists a continuous map $\mathcal{G}(V) \to \varprojlim \mathcal{G}(C_n)$. On the other hand, we have a continuous map $\Theta : \varprojlim \mathcal{G}(C_n) \to \varprojlim \mathcal{G}(U_{n-1})$ given by restriction. Then Θ is a homeomorphism of quasitopological spaces, with inverse $\Theta^{-1} : \varprojlim \mathcal{G}(U_{n-1}) \to \varprojlim \mathcal{G}(C_{n-1})$ given by restriction again.

The composition $\mathcal{G}(V) \to \varprojlim \mathcal{G}(C_n) \to \varprojlim \mathcal{G}(U_{n-1})$ is also a homeomorphism, by the gluing property of continuous sheaves applied to the open cover $\{U_{n-1}\}$ of U. Hence, $\mathcal{G}(V) \cong \varprojlim \mathcal{G}(C_n)$, as desired. □

Claim 4.36. $\varinjlim_n \mathcal{G}(C_n \setminus U') \cong \mathcal{G}(\mathrm{Op}_U(U') \setminus U')$.

Proof of Claim 4.36: Consider the following chain of homeomorphisms

$$\varinjlim_n \mathcal{G}(C_n \setminus U') \cong \varinjlim_n \varinjlim_{V \supset C_n} \mathcal{G}(V \setminus U') \cong \varinjlim_{V \supset U'} \mathcal{G}(V \setminus U') \cong \mathcal{G}(\mathrm{Op}_U(U') \setminus U')$$

where $V \supset C_n$ varies over all open neighborhoods of C_n in U. □

We now proceed to take limits of $r_{m,n}$, first as $m \to \infty$ and then as $n \to \infty$. By Lemma 4.30, this gives that $\mathcal{G}(U \setminus U') \to \mathcal{G}(\mathrm{Op}_U(U') \setminus U')$ is also a fibration. □

Proposition 4.37. *Let (X, Σ_X) be a stratified space with exactly two strata, $\Sigma_X = \{S, L\}$, with $S < L$, so that $X = \overline{L}$. Let $\mathcal{F}$ be a stratified continuous sheaf on X. Assume the following:*

1. *$\mathcal{F}_S = i_S^* \mathcal{F}_{\overline{S}}$ satisfies the parametric h-principle,*
2. *$\mathcal{F}_L = i_L^* \mathcal{F}_{\overline{L}}$ is flexible, and*
3. *for some $\psi \in \mathcal{F}_S(S)$, $\mathcal{H}_S^L = \mathrm{hofib}(\mathrm{res}_S^L; \psi)$ satisfies the parametric h-principle.*

Then $\mathcal{F} = \mathcal{F}_{\overline{L}}$ satisfies the parametric h-principle.

Proof. Let $\psi \in \mathcal{F}_S(S)$ be as in Definition 4.20. Then by hypothesis there exists a homotopy fiber sequence of continuous sheaves over S, given by

$$\mathrm{hofib}(\mathrm{res}_S^L; \psi) \to \iota_S^* \mathcal{F}_{\overline{L}} \to \mathcal{F}_S.$$

By hypothesis, $\mathcal{F}_S$ and $\mathrm{hofib}(\mathrm{res}_{S,L}; \psi)$ satisfy the parametric h-principle. Hence, by Lemma 4.22 so does $\iota_S^* \mathcal{F}_{\overline{L}}$, i.e. the natural inclusion map $\iota_S^* \mathcal{F}_{\overline{L}} \to (\iota_S^* \mathcal{F}_{\overline{L}})^*$ is a weak homotopy equivalence. Combining this with Lemma 4.25, we conclude that the map $(\iota_S^* \mathcal{F}_{\overline{L}})^* \to \iota_S^*(\mathcal{F}_{\overline{L}}^*)$ is a weak homotopy equivalence. By Remark 4.26 we can in fact choose the homotopy inverse to be $(-) \circ \pi_S$. This implies that the resulting composition map

$$\iota_S^* \mathcal{F}_{\overline{L}} \to (\iota_S^* \mathcal{F}_{\overline{L}})^* \to \iota_S^*(\mathcal{F}_{\overline{L}}^*)$$

is simply the restriction of the diagonal normal construction $\mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{L}}^*$ to S.

Let $\Phi : \mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{L}}^*$ denote the natural morphism in the diagonal normal construction. Then $\Phi|L : \mathcal{F}_L \to \mathcal{F}_L^*$ and $\Phi|S : \iota_S^* \mathcal{F}_{\overline{L}} \to \iota_S^*(\mathcal{F}_{\overline{L}}^*)$ are both weak homotopy equivalences. We would like to "glue" these to a weak homotopy equivalence Φ. To this end, let $U \subset \overline{L}$ be an open subset and $U' := U \cap S$. Using Lemma 4.28, we have fiber squares:

$$\begin{array}{ccc} \mathcal{F}_{\overline{L}}(U) & \longrightarrow & \mathcal{F}_{\overline{L}}(U') \\ \downarrow & & \downarrow \\ \mathcal{F}_{\overline{L}}(U \setminus U') & \longrightarrow & \mathcal{F}_{\overline{L}}(\mathrm{Op}_U(U') \setminus U') \end{array} \qquad \begin{array}{ccc} \mathcal{F}_{\overline{L}}^*(U) & \longrightarrow & \mathcal{F}_{\overline{L}}^*(U') \\ \downarrow & & \downarrow \\ \mathcal{F}_{\overline{L}}^*(U \setminus U') & \longrightarrow & \mathcal{F}_{\overline{L}}^*(\mathrm{Op}_U(U') \setminus U') \end{array}$$

The diagonal normal construction map Φ gives natural maps from each corner of the first diagram to the corresponding corner of the second diagram. Note first that Φ is a weak homotopy equivalence on the top-right and bottom-left corners as $\Phi|L$ and $\Phi|S$ are weak homotopy equivalences of continuous sheaves.

Next, note that

$$\Phi : \mathcal{F}_{\overline{L}}(\mathrm{Op}_U(U') \setminus U') \to \mathcal{F}^*_{\overline{L}}(\mathrm{Op}_U(U') \setminus U')$$

is a direct limit of the weak homotopy equivalences $\Phi|(V \setminus U') : \mathcal{F}_{\overline{L}}(V \setminus U') \to \mathcal{F}^*_{\overline{L}}(V \setminus U')$ indexed by open neighborhoods $V \subset U$ of U' in U. Since homotopy groups commute with direct limit of quasitopological spaces, hence the limiting map is also a weak homotopy equivalence. Thus, Φ is a weak homotopy equivalence on the bottom-right corner as well.

We shall use Theorem 4.31 to conclude the proof. To set up the situation such that the hypotheses of Theorem 4.31 are satisfied, we need to establish that

(1) Φ is a weak homotopy equivalence on the top-right, bottom-left and bottom-right corners. The above paragraph did precisely this.
(2) The right vertical arrows in both commutative diagrams above are fibrations. Lemma 4.33 gives this.

Finally, we apply Theorem 4.31 to "coglue" to a homotopy equivalence $\Phi : \mathcal{F}_{\overline{L}}(U) \to \mathcal{F}^*_{\overline{L}}(U)$. This proves that $\mathcal{F}_{\overline{L}}$ satisfies the parametric h-principle. □

The above proof also gives us a criterion for checking the parametric h-principle for a *continuous* sheaf on the underlying topological space of a stratified space.

Lemma 4.38. *Let X be a stratified space. Let $\mathcal{F}$ be a continuous sheaf on X, such that $i_S^*\mathcal{F}$ is flexible for all (open) strata S, then $\mathcal{F}$ satisfies the parametric $h-$principle.*

Proof. We argue by induction on depth of X. If the stratified space X has depth one, i.e. it is a manifold, then the result follows from Gromov's Theorem 4.12. Let L denote the disjoint union of maximal strata of X, i.e. strata that do not lie on the boundary of other strata. Let $\partial L = X \setminus L$. Applying the inductive hypothesis to ∂L, we conclude that $i^*_{\partial L}\mathcal{F}$ satisfies the parametric $h-$principle. By hypothesis, we have flexibility of $i_L^*\mathcal{F}$, since L is an open stratum in X. As in the proof of Proposition 4.37, we use Theorem 4.31 to glue the homotopy equivalences $i^*_{\partial L}\mathcal{F} \to (i^*_{\partial L}\mathcal{F})^*$ and $\iota_L^*\mathcal{F} \to (\iota_L^*\mathcal{F})^*$ to obtain a homotopy equivalence $\mathcal{F} \to \mathcal{F}^*$. Therefore, $\mathcal{F}$ satisfies the parametric h-principle. □

Remark 4.39. *It is important to distinguish the hypothesis of Lemma 4.38 from the hypothesis of stratumwise flexibility in Definition 3.9 which applies to* stratified sheaves*. Stratumwise flexibility is a condition on the* intrinsic sheaves $\mathcal{F}_S$ *of a stratified sheaf, whereas the hypotheses above are related to flexibility of the* extrinsic sheaves $\iota_S^*\mathcal{F}_{\overline{L}}$*, for $S < L$. The difference between these sheaves lies in the homotopy fiber sheaf $\mathcal{H}^L_S := \mathrm{hofib}(\mathrm{res}^L_S; \psi)$.*

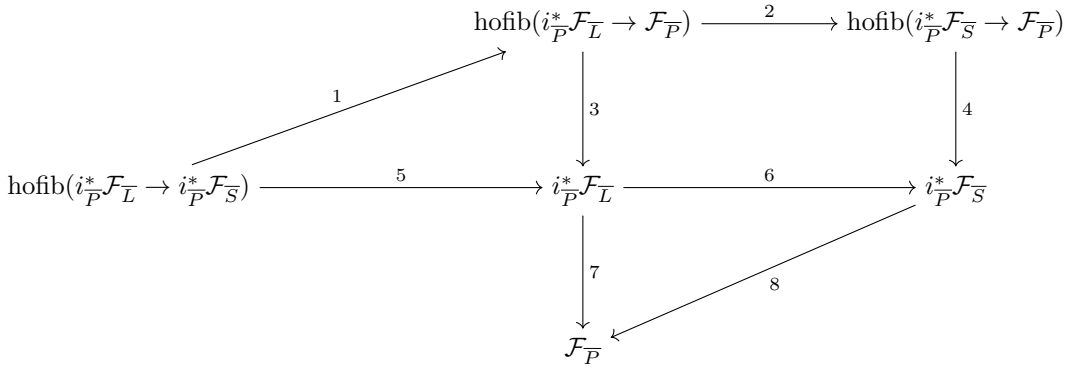
Lemma 4.40. *Let (X, Σ) be a stratified space, and $\mathcal{F}$ be a stratified sheaf on X. For any pair of strata $S < L$ in X, recall the associated closed and open homotopy fiber sheaves $\overline{\mathcal{H}}^L_S$ and $\mathcal{H}^L_S$ from Definition 3.11. For any triad of strata $P < S < L$ in X, there exist homotopy equivalences of sheaves*

$$i_P^*\overline{\mathcal{H}}^L_S \simeq \mathrm{hofib}\left(\overline{\mathcal{H}}^L_P \to \overline{\mathcal{H}}^S_P\right),$$
$$i_P^*\overline{\mathcal{H}}^L_S \simeq \mathrm{hofib}\left(\mathcal{H}^L_P \to \mathcal{H}^S_P\right).$$

Proof. Let $f : X \to Y$ and $g : Y \to Z$. Then there exists a homotopy fiber sequence

$$\mathrm{hofib}(f) \to \mathrm{hofib}(g \circ f) \to \mathrm{hofib}(g)$$

by [MP12, Lemma 1.2.7]. To translate this into the context of continuous sheaves, consider the following diagram:



Here, the arrows 6,7,8 take the place of $f, g \circ f, g$ respectively. The above homotopy theoretic fact ([MP12, Lemma 1.2.7]) then shows that the arrows 1,2 give a homotopy fiber sequence

$$\mathrm{hofib}(i_{\overline{P}}^*\mathcal{F}_{\overline{L}} \to i_{\overline{P}}^*\mathcal{F}_{\overline{S}}) \xrightarrow{1} \mathrm{hofib}(i_{\overline{P}}^*\mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{P}}) \xrightarrow{2} \mathrm{hofib}(i_{\overline{P}}^*\mathcal{F}_{\overline{S}} \to \mathcal{F}_{\overline{P}}).$$

The first term in the sequence above can be identified with $i_{\overline{P}}^*\overline{\mathcal{H}}_S^L$. Indeed,

$$\mathrm{hofib}(i_{\overline{P}}^*\mathcal{F}_{\overline{L}} \to i_{\overline{P}}^*\mathcal{F}_{\overline{S}}) = \mathrm{hofib}(i_{\overline{P}}^* i_{\overline{S}}^*\mathcal{F}_{\overline{L}} \to i_{\overline{P}}^*\mathcal{F}_{\overline{S}}) = i_{\overline{P}}^* \mathrm{hofib}(i_{\overline{S}}^*\mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{S}}) = i_{\overline{P}}^*\overline{\mathcal{H}}_S^L$$

The first statement of the Lemma follows. The second statement follows by replacing $\overline{P}$ by P throughout. □

4.4.2. *Gluing sheaves across strata that intersect along their closure.* In the proof of Theorem 4.42 below, we shall need to glue sheaves in different strata S_1, S_2 to obtain a sheaf over $\overline{S_1} \cup \overline{S_2}$ when $\overline{S_1} \cap \overline{S_2} \neq \emptyset$. Note that Definition 3.8 does not directly furnish such a sheaf. We proceed by induction on height (Definition 2.1). For strata of height zero, there is nothing new to construct. For concreteness, and to illustrate the construction, suppose S_1, S_2 have height one, and $\overline{S_1} \cap \overline{S_2} = P$, where P has height zero. Then there are two extrinsic sheaves $i_P^*\mathcal{F}_{\overline{S_1}}$ and $i_P^*\mathcal{F}_{\overline{S_2}}$, and restriction maps $\mathrm{res}_P^{S_1}, \mathrm{res}_P^{S_2}$ to $\mathcal{F}_P = \mathcal{F}_{\overline{P}}$. Then there exists a natural sheaf $\mathcal{F}_{\overline{S_1} \cup \overline{S_2}}$ given as follows:

(1) On a germinal neighborhood of P in $\overline{S_1} \cup \overline{S_2}$ (where the latter is equipped with the subspace topology inherited from X), $\mathcal{F}_{\overline{S_1} \cup \overline{S_2}}$ is given by the fiber product $i_P^*\mathcal{F}_{\overline{S_1}} \times_{\mathcal{F}_P} i_P^*\mathcal{F}_{\overline{S_2}}$,
(2) On S_1 (resp. S_2), $\mathcal{F}_{\overline{S_1} \cup \overline{S_2}}$ equals $\mathcal{F}_{S_1}$ (resp. $\mathcal{F}_{S_2}$).

The general construction now follows by induction. Assume therefore that for any finite union X_m of strata of height at most m, we have a sheaf $\mathcal{F}_m$ such that

(1) $i_S^*\mathcal{F}_m = \mathcal{F}_S$ for all strata S of height m.
(2) For any stratum closure $\overline{P}$ of height less than m, $\mathcal{F}_m$ equals the fiber product of extrinsic sheaves of the form $i_{\overline{P}}^*\mathcal{F}_{\overline{S}}$, where $\overline{P} \subset \overline{S}$.

Then the gluing construction described above for two strata S_1, S_2 can be repeated for strata of height $m+1$, and the same argument goes through for any finite collection $S_1, \cdots, S_k$ of height $m+1$. In particular, note that this furnishes a well-defined sheaf $\mathcal{F}_Y$ for any closed subset of X that is a union of strata. We note for use below, the following Corollary of the proof of Proposition 4.37.

Corollary 4.41. *Let (X, Σ) be a stratified space with unique top dimensional stratum L. Let $Y = \partial L$. Let $\mathcal{F} = \{\mathcal{F}_{\overline{S}} : S \in \Sigma\}$ be a stratified sheaf on X. Suppose that*

1. *$\mathcal{F}_Y$ satisfies the parametric $h-$principle.*
2. *$\overline{\mathcal{H}}_Y^X := \mathrm{hofib}(i_Y^* \mathcal{F}_X \to \mathcal{F}_Y)$ satisfies the parametric $h-$principle.*
3. *$\mathcal{F}_X|_L$ is flexible.*

Then $\mathcal{F}_X$ satisfies the parametric $h-$principle.

Proof. Lemma 4.22 along with the first two conditions ensure that $i_{\overline{Y}}^* \mathcal{F}_X$ satisfies the parametric $h-$principle. Now, flexibility of $\mathcal{F}_X|L$ allows us to co-glue $\mathcal{F}_X|L$ and $i_{\overline{Y}}^* \mathcal{F}_X$ as in the proof of Proposition 4.37 to conclude that $\mathcal{F}_X$ satisfies the parametric $h-$principle. □

4.4.3. *$h-$principle from stratumwise conditions.* We are now in a position to prove the main theorem of this Section. It says roughly that flexibility of homotopy fibers for pairs of strata guarantees parametric $h-$principle for the stratified sheaf provided the latter is stratumwise flexible. We shall first prove this assuming a total order of strata.

Theorem 4.42. *Let (X, Σ) be a stratified space and $\mathcal{F}$ be a stratified sheaf on X such that*

1. *$\mathcal{F}$ is stratumwise flexible, i.e. for every stratum $S \in \Sigma$, $\mathcal{F}_S := i_S^* \mathcal{F}_{\overline{S}}$ is flexible on the stratum j.*
2. *$\mathcal{F}$ is infinitesimally flexible across strata, i.e. for all $S < L$, the open homotopy fiber sheaf $\mathcal{H}_S^L$ is flexible.*

Then the stratified sheaf $\mathcal{F}$ satisfies the parametric $h-$principle.

Proof of Theorem 4.42 under the assumption of total ordering: We first assume that the strata of X are totally-ordered. Thus,

1. (X, Σ) is a stratified space with strata $\Sigma = \{S_1 > \cdots > S_n\}$ ordered so that $S_i < S_j$ if $i > j$.
2. $\mathcal{F} = \{\mathcal{F}_{\overline{i}} : 1 \leq i \leq n\}$ is a stratified continuous sheaf on X, where $\mathcal{F}_{\overline{i}}$ is a continuous sheaf on the stratum-closure $\overline{S}_i$.

For any pair of indices $i > j$, let $\overline{\mathcal{H}}_i^j = \mathrm{hofib}(i_{\overline{S}_i}^* \mathcal{F}_{\overline{j}} \to \mathcal{F}_{\overline{i}})$, and $\mathcal{H}_i^j = i_{S_i}^* \overline{\mathcal{H}}_i^j$ be the closed and open homotopy fibers, respectively, as in Definition 3.11. Then the hypotheses of the theorem translate to the following:

1. For all j, the sheaf $\mathcal{F}_j := i_{S_j}^* \mathcal{F}_{\overline{j}}$ is flexible on the stratum j.
2. For all $i > j$, the open homotopy fiber sheaf $\mathcal{H}_i^j$ is flexible.

We shall show that the sheaves $\mathcal{F}_{\overline{j}}$ satisfy the parametric $h-$principle. We proceed by induction on n. For $n = 1$, i.e. for manifolds, this is due to Gromov (see the Main Theorem on pg. 76 of [Gro86]).

By induction, we can assume that $\mathcal{F}_{\overline{2}}$ satisfies the parametric $h-$principle (using the chain of $n-1$ strata $S_2 > \cdots > S_n$). We first prove that the closed homotopy fibers $\overline{\mathcal{H}}_j^i$, $i > j$ satisfy the parametric h-principle.

Claim 4.43. *With notation and hypothesis as in Theorem 4.42, $\overline{\mathcal{H}}_i^j$ satisfies the parametric $h-$principle for all $i > j$.*

Proof of Claim 4.43. We prove this by downward induction on i. First, let $i = n$. Since S_n is the deepest stratum and therefore $S_n = \overline{S}_n$, $\overline{\mathcal{H}}_n^j = \mathcal{H}_n^j$, and $\mathcal{H}_n^j$ is flexible by hypothesis. Hence, by [Gro86, p. 76], $\overline{\mathcal{H}}_n^j = \mathcal{H}_n^j$ satisfies the parametric $h-$principle for all $j < n$.

Suppose now that the theorem is true for $i = k+1 \leq n$ and all $j < k+1$. We shall show that $\overline{\mathcal{H}}_k^j$ satisfies the parametric h-principle as a sheaf on $\overline{S}_k$, for all $j < k$. Observe that $i_{S_k}^* \overline{\mathcal{H}}_k^j = \mathcal{H}_k^j$ is an open homotopy fiber, hence flexible by hypothesis. Next, by Lemma 4.40,

$$i_{\overline{S}_{k+1}}^* \overline{\mathcal{H}}_k^j \simeq \text{hofib}\left(\overline{\mathcal{H}}_{k+1}^j \to \overline{\mathcal{H}}_{k+1}^k\right).$$

By the induction hypothesis, $\overline{\mathcal{H}}_{k+1}^j$ and $\overline{\mathcal{H}}_{k+1}^k$ satisfy the parametric h-principle. Therefore, by Lemma 4.22 and Remark 4.23, $i_{\overline{S}_{k+1}}^* \overline{\mathcal{H}}_k^j$ satisfies the parametric h-principle as well. Thus, by Corollary 4.41, $\overline{\mathcal{H}}_k^j$ satisfies the parametric h-principle. □

In particular $\overline{\mathcal{H}}_2^1$ satisfies the parametric h-principle. By Corollary 4.41, we conclude that $\mathcal{F}_{\overline{1}}$ satisfies the parametric h-principle, as required. □

Proof of Theorem 4.42, the general case: We now show how to remove the hypothesis of total orderability of strata in the proof of Theorem 4.42. We shall proceed by induction on height and apply Corollary 4.41. We use the notation of Corollary 4.41. The proof of Corollary 4.41 in fact shows that $\mathcal{F}_X$ satisfies the parametric $h-$principle provided that for *any* maximal (with respect to height) stratum L, and $Y = \partial L$, we can show that $\mathcal{F}_Y$, and $\overline{\mathcal{H}}_Y^X$ satisfy the parametric $h-$principle. The gluing needs flexibility of $\mathcal{F}_L$ in Corollary 4.41, but here our concern will be with the parametric $h-$principle. Let $m+1$ denote the height of L, so that Y has height m. Assume by induction that

(1) $\mathcal{F}_A$ satisfies the parametric $h-$principle for any substratified space $A \subset X$ of height $< m$.
(2) Further, for any substratified space $B \subset X$ with $B > A$, $\overline{\mathcal{H}}_A^B$ satisfies the parametric $h-$principle.

Set $Y = \cup_1^k Y_i$, where Y_i's denote the *closures* of the maximal (with respect to height) strata of Y. By induction on k, it suffices to prove the theorem for $k = 2$. This is because the proof in the totally-ordered case (coupled with the above inductive hypothesis) allows us to conclude that the statement is true for $k = 1$, and we can take the union $\cup_1^{k-1} Y_i$ as a single stratified space in the inductive step. Hence, assume that $Y = Y_1 \cup Y_2$. Also, note that $Z = Y_1 \cap Y_2$ has less height than (at least one of) Y_1, Y_2.

We refer to the sheaf $\mathcal{F}_Y$ (constructed from $\mathcal{F}_{Y_1}$ and $\mathcal{F}_{Y_2}$ as in Section 4.4.2 as the *intrinsic sheaf on* Y. Similarly, we refer to $i_Y^* \mathcal{F}_X$ as the *extrinsic sheaf on* Y. It suffices, by Corollary 4.41 to prove that

(1) The intrinsic sheaf $\mathcal{F}_Y$ satisfies the parametric $h-$principle, and
(2) $\overline{\mathcal{H}}_Y^X = \text{hofib}(i_Y^* \mathcal{F}_X \to \mathcal{F}_Y)$ satisfies the parametric $h-$principle.

$\mathcal{F}_Y$ satisfies the parametric $h-$principle:
To prove that $\mathcal{F}_Y$ satisfies the parametric $h-$principle, it suffices by stratumwise flexibility of $\mathcal{F}$ to show that

$$\mathcal{A} = i_Z^*\mathcal{F}_{Y_1} \times_{\mathcal{F}_Z} i_Z^*\mathcal{F}_{Y_2}$$

satisfies the parametric $h-$principle. (Recall that $Z = Y_1 \cap Y_2$. This reduction is exactly as in the proof Proposition 4.37).

By Lemma 4.32, $\mathrm{hofib}(\mathcal{A} \to i_Z^*\mathcal{F}_{Y_1})$ is homotopy equivalent to $\mathrm{hofib}(i_Z^*\mathcal{F}_{Y_2} \to \mathcal{F}_Z)$. By the inductive hypothesis on height (of Z), $\mathrm{hofib}(i_Z^*\mathcal{F}_{Y_2} \to \mathcal{F}_Z)$ satisfies the parametric $h-$principle. Hence, so does $\mathrm{hofib}(\mathcal{A} \to i_Z^*\mathcal{F}_{Y_1})$. Again, by the inductive hypothesis on height (of Z), $i_Z^*\mathcal{F}_{Y_1}$ satisfies the parametric $h-$principle. Hence, by Lemma 4.22, $\mathcal{A}$ satisfies the parametric $h-$principle.

$\overline{\mathcal{H}}_Y^X$ satisfies the parametric $h-$principle:
We know that the following sheaves satisfy the parametric $h-$principle:

(1) $\mathcal{F}_Z$ and $\mathrm{hofib}(i_Z^*\mathcal{F}_X \to \mathcal{F}_Z)$ (by the inductive hypothesis on closures of strata on lower height, and closed homotopy fibers on closures of strata on lower height)
(2) $\mathrm{hofib}(i_{Y_1}^*\mathcal{F}_X \to \mathcal{F}_{Y_1})$ (from the proof in the totally-ordered case)
(3) $\mathrm{hofib}(i_{Y_2}^*\mathcal{F}_X \to \mathcal{F}_{Y_2})$ (from the proof in the totally-ordered case)

It suffices (as in the proof of Proposition 4.37) to show that $\mathrm{hofib}(i_Z^*\mathcal{F}_X \to \mathcal{A})$ satisfies the parametric $h-$principle.

Note now that by Lemma 4.22, $\mathrm{hofib}(\mathcal{A} \to \mathcal{F}_Z)$ satisfies the parametric $h-$principle (since we have already shown that $\mathcal{A}$ does, and the inductive hypothesis gives that $\mathcal{F}_Z$ does). Further, by the inductive hypothesis applied to the lower depth stratum closure Z, the homotopy fiber $\mathrm{hofib}(i_Z^*\mathcal{F}_X \to \mathcal{F}_Z)$ satisfies the parametric $h-$principle. By Lemma 4.40, $\mathrm{hofib}(i_Z^*\mathcal{F}_X \to \mathcal{A})$ is homotopy equivalent to

$$\mathrm{hofib}\left(\mathrm{hofib}(i_Z^*\mathcal{F}_X \to \mathcal{F}_Z) \to \mathrm{hofib}(\mathcal{A} \to \mathcal{F}_Z)\right).$$

By Lemma 4.22, this satisfies the parametric $h-$principle. Hence, so does $\mathrm{hofib}(i_Z^*\mathcal{F}_X \to \mathcal{A})$. □

4.4.4. *Flexibility versus h-principle.* The aim of the example below is to illustrate the necessity of flexibility of $\mathcal{F}$ on the top stratum in Theorem 4.42 and Proposition 4.37. It emphasizes that it is not enough to assume that $\mathcal{F}$ satisfies the parametric $h-$principle on the top stratum.

Let $X = M$ be an orientable 3-manifold, and $F \subset M$ be an embedded orientable surface. Stratify X with two strata: $L = M \setminus F$ and $S = F$, so that $S = \overline{S}$. Let N be another 3-manifold not covered by M.

The stratified sheaf $\mathcal{F}$ is defined as:

(1) $\mathcal{F}_L(U) = \mathrm{Imm}(U, N)$ for $U \subset \overline{L}$ open,
(2) $\mathcal{F}_S(V) = \mathrm{Imm}(V, N)$ for $V \subset S$ open.

Then the restriction map $\mathrm{res} : i_S^*\mathcal{F}_L \to \mathcal{F}_S$ simply forgets the normal bundle to S in M. We note the following:

(1) $\mathcal{F}_L|L$ satisfies the parametric $h-$principle [Gro86, p. 79] as L is open.
(2) $\mathcal{F}_L|L$ is not flexible (since the dimensions of M, N coincide, and both are compact).
(3) $\mathcal{F}_S$ is flexible [Gro86, p. 79].

(4) hofib(res) is flexible.

Proposition 4.44. *With $M, \mathcal{F}, L, S$ as above, $\mathcal{F}$ does not satisfy the parametric $h-$principle.*

Proof. The Gromov diagonal construction applied to $\mathcal{F}$ gives $\mathcal{F}^*$ homotopy equivalent to a sheaf $\mathcal{G}$ given as follows. $\mathcal{G}(U)$ consists of bundle maps $\psi : TU \to TN$ covering smooth maps $\psi_0 : U \to N$ such that $\psi|T_xU$ is an isomorphism on every tangent space T_xU. In particular, $\mathcal{G}(M)$ consists of bundle maps $\psi : TM \to TN$ covering smooth maps $\psi_0 : M \to N$ such that $\psi|T_xM$ is an isomorphism on every tangent space T_xM. Since M is an orientable 3-manifold, it is parallelizable, so that $TM = M \times \mathbb{R}^3$. Therefore $\mathcal{G}(M)$ contains ψ covering a *constant map* ψ_0. In particular, $\mathcal{G}(M)$ is non-empty.

On the other hand, $\mathcal{F}(M)$ consists of immersions from M to N. Since M, N have the same dimension, $\mathcal{F}(M)$ consists of covering maps. Since N is not covered by M, $\mathcal{F}$ is empty. Hence, $\mathcal{F}$ does not satisfy the parametric $h-$principle. □

4.5. **Microflexibility of stratified sheaves.** For the purposes of this subsection X will denote a stratified space, and V a manifold. $\mathcal{F}$ will be a (stratified) continuous sheaf. The aim of this subsection is to generalize the following theorem of Gromov and its consequence below to stratified spaces and stratified sheaves over them.

Theorem 4.45. [Gro86, p. 78] *Let $Y = V \times \mathbb{R}$, and let $\Pi : Y \to V$ denote the projection onto the first factor. Let $\mathrm{Diff}(V, \Pi)$ be the group of diffeomorphisms of Y commuting with Π, where V is identified with $V \times \{0\}$. Let $\mathcal{F}$ be a microflexible (continuous) sheaf over Y invariant under $\mathrm{Diff}(V, \Pi)$. Then the restriction $\mathcal{F}|V(= V \times \{0\})$ is a flexible sheaf over $V(= V \times \{0\})$.*

The principal consequence is the following flexibility theorem for Diff $-$invariant sheaves.

Theorem 4.46. [Gro86, p. 78-79] *Let $\mathcal{F}$ be a microflexible $\mathrm{Diff}(V)-$invariant (continuous) sheaf over a manifold V. Then the restriction to an arbitrary piecewise smooth polyhedron $K \subset V$ of positive codimension, $\mathcal{F}|K$, is a flexible sheaf over K.*

Recall that a stratified (continuous) sheaf $\mathcal{F}$ over a stratified space X is *stratumwise microflexible* if for every stratum S of X, $\mathcal{F}|S$ is microflexible.

The main theorems of this section are now given below.

Theorem 4.47. *Let $Y = X \times \mathbb{R}$ equipped with the product stratification, and let $\Pi : Y \to X$ denote the projection onto the first factor. Let $\mathcal{F}$ be a stratumwise microflexible continuous sheaf over Y invariant under $\mathrm{StratDiff}(X, \Pi)$. Further, suppose that $\mathcal{F}$ is infinitesimally microflexible across strata, i.e. for all strata $S < L$ of Y, $\mathcal{H}^L_S$ (cf. Definition 3.11) is microflexible. Then the restriction $\mathcal{F}|X(= X \times \{0\})$ is a stratified sheaf over $X(= X \times \{0\})$ satisfying the parametric $h-$principle.*

Proof. Note first that $\mathcal{H}^L_S$ is invariant under $\mathrm{StratDiff}(X, \Pi)$ by Lemma 3.16. By hypothesis,

(1) $\mathcal{F}|L \times \mathbb{R}$ is microflexible.
(2) $\mathcal{H}^L_S$ is microflexible.

The strata L of Y are of the form $L_X \times \mathbb{R}$, where $L_X = L \cap X$. Note that L is a manifold. Invariance of $\mathcal{F}$ and $\mathcal{H}^L_S$ under $\mathrm{StratDiff}(X, \Pi)$ implies $\mathrm{Diff}(L_X \times \mathbb{R}, \Pi)-$invariance of $\mathcal{F}|L_X \times \mathbb{R}$ and $\mathcal{H}^L_S|S_X \times \mathbb{R}$. It follows from Theorem 4.45

that $\mathcal{F}|S_X$ and $\mathcal{H}^L_S|S_X$ are flexible. Hence, by Theorem 4.42, $\mathcal{F}|X$ satisfies the parametric $h-$principle. □

Definition 4.48. *A stratified subspace $K < X$ is said to be of positive codimension if for every stratum S of X, $K \cap S$ has positive codimension in S.*

Theorem 4.49. *Let $\mathcal{F}$ be a stratumwise microflexible* StratDiff $-$*invariant stratified (continuous) sheaf over X. Further, suppose that $\mathcal{F}$ is infinitesimally microflexible across strata, i.e. the open homotopy fiber sheaves $\mathcal{H}^L_S$ (Definition 3.11) are microflexible. Then the restriction $\mathcal{F}|K$ to a stratified subspace $K \subset X$ of positive codimension satisfies the parametric $h-$principle.*

Proof. Since K is of positive codimension in X, for every $k \in K$, there is an open neighborhood U_k of k in K such that $U_k \times (-1, 1)$ embeds in X. The theorem now follows from Theorem 4.47 since locally flexible sheaves are flexible (by Gromov's localization lemma [Gro86, p. 79]). □

Remark 4.50. *Theorems 4.47 and 4.49 provide examples of how to translate a positive codimension stratumwise microflexibility hypothesis into an $h-$principle conclusion.*

5. The Gromov diagonal normal construction for smooth stratified spaces

We specialize the Gromov diagonal normal sheaf construction of $\mathcal{F}^*$ in Definition 4.1 to the sheaf of sections of a stratified bundle E over a smooth stratified space X. Even in the manifold setup, an explicit connection between Gromov's $\mathcal{F}^*$ construction and the use of jets in [EM02] is a little difficult to find. Hence, we provide a detailed treatment below. Remark 4.4, which gives an explicit description of $\mathcal{F}^*$ will allow us to formalize this. It will turn out that in the stratified context, $\mathcal{F}^*$ admits an inductive description up to homotopy in terms of two constituent sheaves:

(1) A purely topological germ of sections (see Definition 5.7 below).
(2) A smooth jet $\mathcal{J}^r_E$ when E is a smooth bundle over a manifold (see Proposition 5.2 below).

The aim of this section is to describe this structure of $\mathcal{F}^*$. In the process we answer Sullivan's question 1.2.

5.1. Tangent microbundles on stratified spaces.

The stratified tangent bundle (Definition 2.22) will turn out to be a stratified subbundle of the tangent microbundle to X (Definition 4.2).

Note that for X a manifold, the tangent microbundle is germinally equivalent to the tangent bundle TX, as it coincides with the normal bundle to the diagonal $\mathrm{diag}(X) \subset X \times X$. For each stratum S of X, TS will thus refer to the tangent microbundle of the manifold S, i.e. it may be identified canonically with the germ of the zero-section from S to the usual (manifold) tangent bundle of S.

The tangent microbundle to a stratified space X, denoted by tX henceforth, turns out to be a stratumwise bundle (see Definition 2.20). We provide an explicit description of tX in terms of the local structure of X.

Let $tX := (U, X, p)$ be the tangent microbundle of X. For $x \in X$, consider the fiber $p^{-1}(x)$. This is a germ U_x of a neighborhood of x in X. Let S be the unique

stratum of X containing x. Then there is an identification of U_x with $W \times cA$ where W is the germ of a ball around x in S and cA is the germ of a cone on the link A of S in X, with cone point x. Thus,

$$(p^{-1}(x), \{(x,x)\}) \cong (U_x, \{x\}) \cong (W, \{x\}) \times (cA, \{x\})$$

as germs of spaces. The bundle over S whose fibers are (germs of) the cones cA will be denoted as NS, and referred to as the normal cone microbundle of S in X.

Hence the restriction of (U, p) to S, i.e. $(p^{-1}(S) \cap U, S, p)$ is germinally equivalent to the direct sum, i.e. fiberwise product of the microbundles:

$$(p^{-1}(S) \cap U, S, p) \cong (TS, S, p) \oplus (NS, S, p),$$

where TS is the tangent microbundle of S and (NS, S, p) is the normal cone microbundle of S in X. This demonstrates that tX is a stratumwise fiber bundle over X according to Definition 2.20, where the fiber over a point $x \in S$ of any particular stratum S of X is given by $T_xS \oplus cA_S(x)$ where A_S denotes the link of S in X, and $cA_S(x)$ denotes the normal cone of S in X at the point x. We next describe a filtration of tX, which induces the canonical filtration of any normal cone cA_S by stratum-closures.

Relative tangent microbundle: Next, suppose that L is a stratum of X and let $Y = \overline{L}$ be the stratum closure of L. Then Y is stratified naturally by the strata of X given by the union of L and those strata of X that lie on the boundary of L. The tangent microbundle for Y can be constructed as above, replacing X by Y. We denote the tangent microbundle tY of Y by $t(L; X)$ and call it the *relative tangent microbundle (relative to L)*. Note that $t(L; X)$ is a microbundle over Y. If, moreover, L is a dense stratum in X, then $t(L; X) = tX$.

Filtering the tangent microbundle: Observe that tX admits a filtration by $t(L; X)$ for L varying over strata of X. Thus, $t(L; X)$ is a sub(micro)bundle of tX restricted to any stratum $S < L$, as

$$t(L; X)|_L = TS \oplus N_{\bar{L}}(S)$$

and

$$tX|_S = TS \oplus N_X(S),$$

where $N_{\bar{L}}(S)$ and $N_X(S)$ denotes the normal cone microbundles of the stratum S in $\bar{L}$ and X, respectively. So $t(L; X)|_S \subset t(X)|_S$. For any particular stratum S of X, the collection $\{t(L; X) : S < L\}$ induces a filtration of the normal bundle $N_X(S)$ by the subbundles $\{N_{\bar{L}}(S) : S < L\}$. These in turn induce the filtration by stratum-closures $\{cA_S^L : S < L\}$ on any particular conical fiber cA_S. Here A_S^L denotes the link of S in $\bar{L}$.

5.2. **Gromov diagonal normal construction for manifolds.** We detail some of the points made in Remark 4.4 and briefly recall Gromov's diagonal normal sheaf construction in the manifold context before generalizing to stratified spaces. Recall (Definition 4.1) that if $\mathcal{F}$ is a continuous sheaf over a manifold M, then the sheaf $\mathcal{P}$ over $M \times M$ is given by $\mathcal{P}(U \times V) = \mathrm{Maps}(U, \mathcal{F}(V))$. Further, the Gromov diagonal normal sheaf $\mathcal{F}^*$ is obtained by restricting $\mathcal{P}$ to the diagonal $\mathrm{diag}(M) \subset M \times M$.

Let $p : E \to M$ be a smooth fiber bundle over M and $\mathcal{F}$ be the sheaf of sections of E, i.e. $\mathcal{F}(U) = \Gamma(U; E)$, where the space of sections $\Gamma(U; E)$ is equipped with the quasitopology inherited from $\mathrm{Maps}(U, E)$. Then

$$\mathcal{P}(U \times V) = \mathrm{Maps}(U, \mathcal{F}(V)) = \mathrm{Maps}(U, \Gamma(V; E)).$$

Therefore, these consist of those maps $U \times V \to E$ for which the restriction to $\{u\} \times V$ gives a section of E over V, for any $u \in U$.

Recall that a collection of elements $\phi_i \in \mathcal{P}(U_i \times U_i), i = 1, \cdots, m$ is consistent if for all $i \neq j$, $\phi_i = \phi_j$ on $(U_i \cap U_j) \times (U_i \cap U_j)$. Then $\mathcal{F}^*$ can be described as follows. For any open set $W \subset \mathrm{diag}(M)$, an element of $\mathcal{F}^*(W)$ consists of a collection of consistent elements $\phi_i \in \mathcal{P}(U_i \times U_i)$ where $\{U_i \times U_i\}$ is a basic open cover of W in $M \times M$. Consistency of ϕ_i ensures that they define a well-defined germ of a map $(TW, W_0) \to E$ at the zero-section W_0 of the tangent bundle $TW \subset TM$.

Let M_0 denote the zero-section of TM. Then, (global) sections of $\mathcal{F}^*$ may be viewed as certain germs of maps $\psi : (TM, M_0) \to E$. The fact that the maps $U_i \times U_i \to E$ are sections when restricted to the second factor implies the following two facts about the germ $\psi : (TM, M_0) \to E$:

(1) ψ is a section $M \to E$ when restricted to the 0-section M_0 of TM.
(2) ψ is a germ of a section $(T_pM, 0) \to E$ when restricted to the tangent space $T_pM \subset TM$ at $p \in M$.

Replacing M by U, $\mathcal{F}^*(U)$ consists of germs of mappings $\psi_U : (TU, U_0) \to E$ such that ψ_U is a section of E when restricted to the 0-section $U_0 \subset TU$ and is a germ of a section of E over a neighborhood of p when restricted to any tangent space T_pU for $p \in U$. Thus, any element of $\mathcal{F}^*(U)$ consists of a section s of E over U decorated with a collection of germs $g_p : (U_p, p) \to (E, s(p))$ of sections of E. Here,

(1) g_p is defined on some neighborhood U_p of $p \in U$, and sends p to $s(p)$.
(2) p ranges over all U.

Therefore, we shall henceforth denote an element of $\mathcal{F}^*(U)$ as a tuple $(s, \{g_p : p \in U\})$. It is convenient to imagine this data as a *base* section $s : U \to E$ of E with the image $s(U) \subset E$ decorated by a *field* of section-germs $\{g_p : p \in U\}$.

There exists a natural morphism of sheaves $\Psi_r : \mathcal{F}^* \to \mathcal{J}^r_E$ from $\mathcal{F}^*$ to the sheaf of $r-$jets of sections of E over M, essentially given by setting $\Psi_r(s, \{g_p\})$ equal to the family of r-order Taylor polynomials of g_p at p, as p varies over U. We state a precise definition below:

Definition 5.1. *For any $(s, \{g_p : p \in U\}) \in \mathcal{F}^*(U)$, define $\Psi_r(s, \{g_p\}) \in \mathcal{J}^r_E(U)$ to be the section of the r-jet bundle of E such that at the point $p \in U$, the section takes the value $J^r_p g_p$. This defines a morphism of sheaves $\Psi_r : \mathcal{F}^* \to \mathcal{J}^r_E$.*

Proposition 5.2. *$\Psi_r : \mathcal{F}^* \to \mathcal{J}^r_E$ is a weak homotopy equivalence of sheaves. Equivalently, for any r, the Gromov diagonal normal sheaf $\mathcal{F}^*$ is naturally homotopy equivalent to the sheaf $\mathcal{J}^r_E$ of $r-$jets of sections of E.*

Proof. Consider the space $C^\infty_0(\mathbb{R}^n, \mathbb{R}^m)$ of germs of smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ at the origin. Also, let $P^r_0(\mathbb{R}^n, \mathbb{R}^m) \subset C^\infty_0(\mathbb{R}^n, \mathbb{R}^m)$ denote the subspace of germs of polynomials (in n variable) of degree at most r at 0. Let $T_r(f)$ denote the Taylor expansion of $f \in C^\infty_0(\mathbb{R}^n, \mathbb{R}^m)$ at 0, truncated at degree r. Then

$$f_t := T_r(f) + t(f - T_r(f)), t \in [0, 1]$$

furnishes a deformation retraction of $C^\infty_0(\mathbb{R}^n, \mathbb{R}^m)$ onto $P^r_0(\mathbb{R}^n, \mathbb{R}^m)$. As locally $\mathcal{F}^*$ and $\mathcal{J}^r_E$ are isomorphic, respectively, to the sheaves $\mathrm{Maps}(-, C^\infty_0(\mathbb{R}^n, \mathbb{R}^m))$ and $\mathrm{Maps}(-, P^r_0(\mathbb{R}^n, \mathbb{R}^m))$, we conclude Ψ_r is a weak homotopy equivalence. □

Remark 5.3. *In fact, $\Psi_\infty : \mathcal{F}^* \to \mathcal{J}^\infty_E$, sending $(s, \{g_p : p \in U\}) \in \mathcal{F}^*(U)$ to its infinite jets is also surjective. If we restrict only to analytic sections, then Ψ_∞ is moreover injective.*

$\Psi_1 : \mathcal{F}^* \to \mathcal{J}^1_E$ given by

$$\Psi_1(s, \{g_p\}) = (s, \{dg_p\})$$

is of particular significance. It replaces the germ-field $\{g_p : p \in U\}$ decorating the base section s by the tangent plane fields $\{dg_p : p \in U\}$.

5.3. **Gromov diagonal normal construction for cones.** We would like to extend the linearized notion of formal $r-$jets of sections ensured by Definition 5.1 and Proposition 5.2 from the manifold context to the context of a stratified bundle $P : E \to X$ over a stratified space. However, a full linearization is not possible and we shall provide a hybrid construction, interbreeding

(1) the linear structure within manifold strata provided by Definition 5.1 and Proposition 5.2,
(2) the germ construction in Remark 4.4 for cones on links using the local structure given by Corollary 2.18.

In this subsection, we shall focus on the second ingredient, and in the next subsection indicate how to assemble these two together. Let $P : E \to X$ be a stratified bundle. For the purposes of this subsection, $E = cB, X = cA$, where $E = B \times [0,1)/B \times \{0\}$, and $X = A \times [0,1)/A \times \{0\}$ and A, B are compact abstractly stratified spaces. Let c_A (resp. c_B) denote the cone-point of cA (resp. cB). Let cB^0 (resp. cA^0) denoted the deleted cone $cB \setminus \{c_B\}$ (resp. $cA \setminus \{c_A\}$).

By Corollary 2.18, there exists a stratified bundle $p : B \to A$ such that (after reparametrization if necessary), $P(b,t) = (p(b), t)$. Hence, there exists a natural stratified bundle $P^0 : cB^0 \to cA^0$ induced by P.

Let $\mathcal{F}$ (resp. $\mathcal{F}_w$) denote the sheaf of controlled (resp. weakly controlled) sections of $P : E \to X$ in this case. Note that any section of $P : cA(= E) \to cB(= X)$ necessarily sends c_A to c_B. Let $\mathcal{F}^0$ (resp. $\mathcal{F}^0_w$) denote the induced sheaf of controlled (resp. weakly controlled) sections of $P^0 : cB^0 \to cA^0$.

We shall first inductively describe the Gromov diagonal normal sheaf for $\mathcal{F}^0$ (resp. $\mathcal{F}^0_w$) using the sheaf $\mathcal{L}$ (resp. $\mathcal{L}_w$) of controlled (resp. weakly controlled) sections of $p : B \to A$ and Lemma 3.13. Let $\mathcal{L}^*$ (resp. $\mathcal{L}^*_w$) denote the Gromov diagonal normal sheaves of $\mathcal{L}$ (resp. $\mathcal{L}_w$). In particular, when A, B are manifolds, then (see Section 5.2):

(1) $p : B \to A$ is a smooth bundle map,
(2) $\mathcal{L} = \mathcal{L}_w$ is the sheaf of smooth sections given by $\mathcal{L}(U) = \Gamma(U, B)$
(3) $\mathcal{L}^*$ is homotopy equivalent to the sheaf $\mathcal{J}^r_B$ of $r-$jets (Proposition 5.2).

Then controlled sections of $P^0 : cB^0 \to cA^0$ are given by maps of the form $\sigma : cA^0 \to cB^0$ of the form $\sigma(a,t) = (s_t(a), t)$, where $s_t : A \to B$ is a controlled section of $p : B \to A$, i.e. σ is a continuous $(0,1)-$parametrized family of sections from A to B. The same holds for controlled sections of $P^0 : (P^0)^{-1}(U) \to U$ for all open U in cA^0. Therefore, we have an isomorphism of sheaves $\mathcal{F}^0 \cong \mathrm{Maps}^p((0,1), \mathcal{L})$, where Maps^p denotes the parametric sheaf as defined in Definition 4.7. The following is now an immediate consequence of Lemma 4.9:

Lemma 5.4. *With $\mathcal{F}^0, \mathcal{L}$ as above, and open subsets $V \subset A$, and $W \subset (0,1)$, we have:*

$$(\mathcal{F}^0)^*(W \times V) = \mathrm{Maps}(W, \mathcal{L}^*(V)).$$

Next, we describe the relationship between the sheaf of controlled sections and the sheaf of weakly controlled sections. Suppose $X = cA$ is equipped with a control structure (i.e. both a projection π and a radial function ρ in a neighborhood of c_A). Let ρ_A denote the radial function on a small neighborhood of c_A in cA. Without loss of generality, by shrinking cA if necessary, we may assume that $\rho_A : cA^0 \to (0,1)$ is a fiber bundle with fiber A. Pulling back ρ_A under P we obtain a radial function $\rho_B = P \circ \rho_A$ on cB.

Assumption 5.5. *Thus, without loss of generality, we assume that $\rho_B = P \circ \rho_A$, i.e. P is a controlled stratified bundle map from cB to cA in a neighborhood of c. Further, $P^0 : cB^0 \to cA^0$ is a bundle map such that*

$$\rho_B(t, b) = t = \rho_A(P^0(t, b)).$$

We shall say that a section $s : cA^0 \to cB^0$ is *levelwise weakly controlled* if $s(\{t\} \times A^0) \subseteq \{t\} \times B^0$ for all $t \in (0,1)$ and the restriction $s|(\{t\} \times A^0)$ is a weakly controlled map to $\{t\} \times B^0$. With the control structure on the domain and target of $P : cB \to cA$ in place, the space of weakly controlled sections $\Gamma_w(cA^0, cB^0)$ fibers over the space of levelwise weakly controlled sections $\Gamma_\ell(cA^0, cB^0)$. The fibers of this fibration are given by reparametrizing the $(0,1)-$direction in cA^0. More precisely, there exists a surjection $\Theta : \Gamma_w(cA^0, cB^0) \to \Gamma_\ell(cA^0, cB^0)$, such that for any $\sigma \in \Gamma(cA^0, cB^0)$, $\Theta^{-1}(\sigma) \cong \mathrm{Maps}(A, \mathrm{Diff}^+((0,1)))$, where $\mathrm{Diff}^+((0,1))$ denotes the orientation-preserving diffeomorphisms of $(0,1)$. Thus, there is a natural product fibration:

$$\Gamma_w(cA^0, cB^0) = \Gamma_\ell(cA^0, cB^0) \times \mathrm{Maps}(A, \mathrm{Diff}^+((0,1)))$$

This is because $\Gamma_w(cA^0, cB^0)$ is a principal $\mathrm{Maps}(A, \mathrm{Diff}^+((0,1)))-$bundle over $\Gamma_l(cA^0, cB^0)$ equipped with a natural section given by the inclusion $\Gamma_\ell(cA^0, cB^0) \hookrightarrow \Gamma_w(cA^0, cB^0)$ of levelwise weakly controlled sections into the weakly controlled sections. Since $\mathrm{Diff}^+((0,1))$ is contractible, we obtain a homotopy equivalence between $\Gamma_w(cA^0, cB^0)$ and $\Gamma_\ell(cA^0, cB^0)$. Consequently, we obtain

Corollary 5.6. *With $\mathcal{F}^0_w, \mathcal{L}_w$ as above, and open subsets $V \subset A$, and $W \subset (0,1)$, we have*

$$(\mathcal{F}^0_w)^*(W \times V) = \mathrm{Maps}(W, (\mathcal{L}_w)^*(V)).$$

Definition 5.7. *The space of* germs *of controlled (resp. weakly controlled) sections of $P : cB(= E) \to cA(= X)$ will be denoted by $\Gamma_c(cA, cB)$ (resp. $\Gamma_{c,w}(cA, cB)$).*

We are now in a position to note the following Proposition which allows us to assemble the descriptions in Lemma 5.4 and Definition 5.7. This is useful in providing an inductive description of the Gromov diagonal normal sheaf of sections of a stratified bundle.

Proposition 5.8. *Any element of the sheaf $\mathcal{F}^*(U)$ (resp. $\mathcal{F}^*_w(U)$) for an open $U \subset cA$ determines and is determined by the following:*

1. *a controlled (resp. weakly controlled) section s over U. In particular, if $U = cA$, $s : cA \to cB$ is a global controlled (resp. weakly controlled) section,*
2. *a germ at c_A given by an element of $\Gamma_c(cA, cB)$ (resp. $\Gamma_{c,w}(cA, cB)$) if $c_A \in U$,*

(3) *an element of* $(\mathcal{F}^0)^*$ *(resp.* $(\mathcal{F}^0_w)^*$*) whose first coordinate (as in Remark 4.4) coincides with the restriction* $s|_{U\setminus\{c_A\}}$.

Proof. For concreteness, we work with $\mathcal{F}$ and $\mathcal{F}^*$. The same argument works for $\mathcal{F}_w$ and $\mathcal{F}^*_w$. Further, Equation 5.3 really ensures that up to homotopy, these sheaves are the same, as we can apply induction (on depth) to weakly controlled sections from A to B in Equation 5.3.

We use the description of the Gromov diagonal normal sheaf from Remark 4.4: any element of $\mathcal{F}^*(U)$ consists of a controlled section s over U decorated with germs of sections $\{g_w : w \in U\}$. The controlled section s over U contributes item 1 in the statement. Next,

(1) For $w = c_A$, g_w is given by an element as in item 2 in the statement.
(2) For $U' = U \setminus \{c_A\}$, $\{g_w : w \in U'\}$ constitutes an element as in item 3 in the statement.

Finally, we observe that the choices in items 2, 3 are independent. Hence any choice as in items 2, 3 subject to the choice of a section s as in item 1 furnishes an element of $\mathcal{F}^*$. □

Proposition 5.8 allows us to decompose elements of $\mathcal{F}^*$ into two independent components. Item 2 provides a purely topological component of $\mathcal{F}^*$. This component cannot be linearized to germs in general. Item 3 on the other hand is defined inductively, and is decomposable, albeit implicitly. Hence Item 3 is again a hybrid of objects as in Item 2 and Item 3, where the latter has 'less complexity of nonlinear objects'. The case where A, B are manifolds is the lowest complexity case. In this case, Proposition 5.2 provides a completely linear description as a sheaf of jets (linearized germs). We note, however, that this last linear description is true only up to homotopy equivalence.

Note also that for manifolds, elements in $\mathcal{F}^*(U)$ may be identified with U-parametrized sections from the tangent space $T_pU \to E$ provided there is a way (e.g. a connection) of identifying T_pU and T_qU for all $p, q \in U$. Regarding the tangent bundle TU as the germ of a neighborhood of the diagonal $\operatorname{diag} U \subset U \times U$, elements in $\mathcal{F}^*(U)$ are thus equivalent to $U-$parametrized sections of E over the normal space $N_{(p,p)}(\operatorname{diag} U)$ to the diagonal at some point $(p, p) \in \operatorname{diag} U$.

Suppose $\mathcal{F}$ is a sheaf of topological spaces over a manifold M such that the inclusion $\mathcal{F} \subset \mathcal{F}^*$ has an inverse given by a retraction of sheaves $r : \mathcal{F}^* \to \mathcal{F}$ so that r is a fibration. Let $\mathcal{P}$ denote the sheaf over $M \times M$ given in Section 5.2. Define a stratification of $X = M \times M$ with strata $S = \operatorname{diag} M$ and $L = (M \times M) \setminus S$. Then we define a stratified sheaf $\mathcal{R}$ over X so that

(1) $\mathcal{R}|\overline{L} = \mathcal{P}$
(2) $\mathcal{R}|S = \mathcal{F}$
(3) the restriction map from $\mathcal{R}|\overline{L}$ to $\mathcal{R}|S$ is given by first restricting $\mathcal{P}$ to S, to obtain $\mathcal{F}^*$, and then composing with the fibration r.

Then (Definition 3.12) $\mathcal{R}$ is infinitesimally flexible across strata.

5.4. **Gromov diagonal normal construction: general case.** For the purposes of this subsection, $(E, \Sigma_E, \mathcal{N}_E)$ and $(X, \Sigma_X, \mathcal{N}_X)$ are abstractly stratified spaces (see Definition 2.6 for notation) and $P : E \to X$ is a stratified fiber bundle. Then, Lemma 2.15 and Corollary 2.18 give us the following commutative diagram.

$$\begin{array}{ccccc} cB & \longrightarrow & \widetilde{N} & \xrightarrow{\widetilde{\pi}} & \widetilde{S} \\ {\scriptstyle P}\downarrow & & {\scriptstyle P}\downarrow & & {\scriptstyle P}\downarrow, \\ cA & \longrightarrow & N & \xrightarrow{\pi} & S \end{array}$$

where the horizontal rows are fiber bundles.

Recall (Remark 4.4) that a formal section of $P : E \to X$ on an open subset $U \subset X$ is a germ of a continuous map $s^* : \mathrm{Op}_{U\times U}(\mathrm{diag}(U)) \to E$ from the germ of an open neighborhood $\mathrm{Op}_{U\times U}(\mathrm{diag}(U))$ of the diagonal $\mathrm{diag}(U) \subset U \times U$ such that

(1) $s : U \to E$ defined by $s(u) := s^*(u, u)$ is a section of E over U.
(2) For every $u \in U$, $s_u : \mathrm{Op}_U(u) \to E$ defined by $g_u(v) = s(u, v)$ is a germ of a section of E in $\mathrm{Op}_U(u) \in U$.
(3) For every stratum $S \in \Sigma_X$ of X intersecting U, s is smooth on $S' = S \cap U$, i.e. $s^*|\,\mathrm{Op}_{S'\times S'}(\mathrm{diag}(S'))$ is a smooth germ of a map to the unique manifold stratum $\widetilde{S} \subset E$ containing $s(S)$.

Henceforth, in this subsection, we shall refer to s as the *base* of the formal section.

Definition 5.9. *A formal section s^* is called a* holonomic section *of $P : E \to X$ over U if*

$$s^*|\,\mathrm{Op}(u) = g_u$$

for all $u \in U$.

Let s^* be a formal section of $P : E \to X$ over X. For every stratum S, we can restrict s^* to $\mathrm{Op}_{S\times X}(\mathrm{diag}(S))$. Note that $\mathrm{Op}_{S\times X}(\mathrm{diag}(S))$ is isomorphic as a microbundle to

$$(S \times S, p_1, \mathrm{diag}(S)) \oplus (N_S, \pi_S, 0_S),$$

where N_S is the normal neighborhood of S (cf. Definition 2.6) and 0_S is naturally identified with $S \subset N_S$. Here, N_S is thought of as the (micro)normal bundle to S in X with fiber cA, where A is the link of S (see, for instance, the commutative diagram above).

Definition 5.10. *Using the microbundle-isomorphism*

$$\mathrm{Op}_{S\times X}(\mathrm{diag}(S)) \cong (S \times S, p_1, \mathrm{diag}(S)) \oplus (N_S, \pi_S, 0_S),$$

(1) *restricting s^* to the first component, we obtain the* tangential formal section *$s^*_{S,t} : \mathrm{Op}_{S\times S}(\mathrm{diag}(S)) \to E$, with base section $s_{S,t}$;*
(2) *restricting s^* to the second component, we obtain the* normal formal section *$s^*_{S,n} : \mathrm{Op}_{N_S}(0_S) \to E$, with base section $s_{S,n}$.*
(3) *$s^*_{S,n}$ and $s^*_{S,t}$ have the same base section $s^*_{S,n}|0_S = s^*_{S,t}|\,\mathrm{diag}(S) = s_S$.*

Remark 5.11 (Normal formal is holonomic)**.** Restriction of the normal formal section $\mathrm{s}^*_{S,n} : \mathrm{Op}_{N_S}(0_S) \to E$ to the fiber $c_xA = cA(x) \subset N_S$ of the normal bundle over a point $x \in S$ is a germ of a controlled section $\mathrm{s}^*_{S,n}|\,\mathrm{Op}_{cA(x)}(x)$ of the conical component of the stratified bundle $(\mathbf{I}, P_2) : cB \to cA$ near the cone point $\{c_x\} \subset cA$. Thus, for an open chart $V \subset S$ around x, we obtain a map:

$$\phi_V : V \to \Gamma_c(cA, cB),\ \phi(x) := \mathrm{s}^*_{S,n}|\,\mathrm{Op}_{cA(x)}(x)$$

On the other hand, the restriction of the normal formal section $\mathrm{s}^*_{S,n}$ to the zero section $0_S \subset N_S$ of the normal bundle returns the base s_S of the formal section. Therefore, $\mathrm{s}^*_{S,n}|_V$ is a germ of a *holonomic section* around $V \times cA \cong \pi_S^{-1}(V) \subset N_S$,

$$(\mathrm{s}_S|_V, \phi_V) : \mathrm{Op}_{V\times cA}(V \times 0) \to E.$$

We pause to emphasize that while this says that $\mathrm{s}^*_{S,n}$ is globally a (germ of a) holonomic section of E over N_S, there is not a meaningful way to write $\mathrm{s}^*_{S,n}$ as a section of E over S (namely, s_S), together with a $S-$parametrized family of sections in $\Gamma_c(cA, cB)$. For one, observe that the bundle homomorphism $P : P^{-1}(N_S) \to N_S$ does not induce a unique map $(\mathbf{I}, P_2) : cB \to cA$ between the normal conical fibers, but rather a unique equivalence class of such maps under $\mathrm{Homeo}_c(cA)-$ and $\mathrm{Homeo}_c(cB)-$valued cocycles acting on the domain and range, respectively.

In general it is not possible to recover the germ $s^*|\operatorname{Op}_{S\times X}(\mathrm{diag}(S))$ from the tangential and normal formal sections. However, for any $\varepsilon > 0$, $s^*|\operatorname{Op}_{S\times X}(\mathrm{diag}(S))$ is ε-close to ${s^*}_{S,t} \oplus {s^*}_{S,n}$ by continuity in the C^0-norm. Here,

$$ {s^*}_{S,t} \oplus {s^*}_{S,n}(x, y, z) = ({s^*}_{S,t}(x, y), {s^*}_{S,n}(x, z)) \in \widetilde{N}_S \subset E $$

for $x \in S$, $y \in \operatorname{Op}_S(x)$, $z \in \operatorname{Op}_{cA(x)}(x)$, where $cA(x) = \pi^{-1}(x)$ (cf. commutative diagram above). For Definition 5.12 below, we assume that X, E are equipped with a metric as at the end of Section 2.4. Further, when we say that two formal sections are ε-close, it is in the sense of closeness with respect to such a metric.

Definition 5.12. *Let $s^* : \operatorname{Op}_{X\times X}(\mathrm{diag}(X)) \to E$ be a formal section of $P : E \to X$ over X. Let $S < L$ be a pair of strata in X. The $\delta-$neighborhood of S in L will be denoted as $N_\delta(S, L)$. We shall say that s^* is of* regularity C^r *if for all pairs $S < L$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

1. *$s^*_{S,n}|N_\delta(S, L) : N_\delta(S, L) \to E$ is smooth on the open stratum L,*
2. *$s^*_{S,t} \oplus (s^*_{S,n}|N_\delta(S, L))$ is ε-close to $s^*_{L,t}$ in the C^r norm. We shall summarize this condition by saying that $s^*_{L,t}$ is C^r-*asymptotic *to $s^*_{S,t} \oplus (s^*_{S,n}|N_\delta(S, L))$.*

The sheaf of C^r-regular holonomic (resp. formal) sections over X will be denoted as $\mathcal{F}_r$ (resp. $\mathcal{F}^*_r$). Let $W \subset X$ be open equipped with the inherited stratification.

Definition 5.13. *For every stratum $S \subset W$ of W, and a section $s : W \to E$,*

1. *Let A_S be the link of S in X,*
2. *Let $\widetilde{S}$ be the unique stratum of E containing $s(S)$,*
3. *Let B_S be the link of $\widetilde{S}$ in E,*
4. *Let $p = (\mathbf{I}, P_2) : cB_S \to cA_S$ be the restriction of $P : E \to X$.*

Let $r \geq 1$. An element of $\mathrm{Strat}\,\mathcal{J}^r(W)$ *consists of a section $s : W \to E$ decorated by the following data corresponding to every stratum $S \subset W$:*

1. *A normal formal section $s^*_{S,n} : \operatorname{Op}_{N_S}(0_S) \to E$ with base s,*
2. *A formal $r-$jet $\sigma_S \in \mathcal{J}^r_E(S)$ of the fiber bundle $P : P^{-1}(S) \to S$.*

such that the following compatibility condition is satisfied. For every stratum $S \subset W$, consider σ_S as an element of the sheaf of formal sections of E over S. Then, for any pair of strata $S < L$ of W,

$$ \sigma_L \text{ is } C^r-\text{asymptotic to } \sigma_S \oplus s^*_{S,n} $$

We summarize the condition by saying $\{\sigma_S\}$ is normally C^r-compatible.

Proposition 5.14. *For any $r \geq 1$, the sheaf $\mathcal{F}^*_r$ is homotopy equivalent to* $\mathrm{Strat}\,\mathcal{J}^r$.

Proof. Consider the homomorphism of sheaves $\Phi : \mathcal{F}^*_r \to \mathrm{Strat}\,\mathcal{J}^r$ given on an open set $W \subset X$ by $\Phi(W) : \mathcal{F}^*_r(W) \to \mathrm{Strat}\,\mathcal{J}^r(W)$, where

$$ \Phi(W)(s^*) = (s, \{s^*_{S,n}\}, \{J^r(s^*_{S,t})\}) $$

Here, for every stratum $S \subset W$, $s^*_{S,n}$ and $s^*_{S,t}$ denote respectively the normal formal and tangential formal components of s^* along S. Also, $J^r(s^*_{S,t})$ denotes $(s_{S,t}, \{J^r g_p : p \in S\})$ where we use the description $s^*_{S,t} = (s_S, \{g_p : p \in S\})$ of the tangential formal section as a base section on S decorated by a germ-field of sections, as in Section 5.2. This is a well-defined map, by normal C^r-compatibility (Definition 5.13).

The candidate for a homotopy inverse is given by the inclusion $\iota : \operatorname{Strat} \mathcal{J}^r \hookrightarrow \mathcal{F}^*_r$ as a subsheaf, by considering a formal r-jet as a formal section. Observe that $\Phi \circ \iota = \mathrm{Id}$. To demonstrate that $\iota \circ \Phi(W)$ is homotopic to the identity map, we follow the proof of Proposition 5.2. For every stratum S of W, consider the straightline homotopy

$$F_t(s^*) = ts^* + (1-t)(J^r(s^*_{S,t}) \oplus s^*_{S,n}), t \in [0,1]$$

This establishes the desired deformation retract on every stratum. □

Remark 5.15. *Specializing the constructions of this entire section to the case of a manifold with corners, or even more specifically, to a simplex, the inductively defined structure given by Propositions 5.14, 5.8 and 5.2 simplifies considerably, giving families of flags of tangent spaces. This answers Sullivan's question 1.2.*

6. Holonomic approximation theorem and other consequences

6.1. **Flexibility of jet sheaves.** Let $P : E \to X$ be a stratified bundle (Definition 2.14). Let $\mathcal{F}$ and $\mathcal{F}_w$ denote respectively the stratified continuous sheaves of controlled and weakly controlled sections of P. Let $\mathcal{H}^r_E, \mathcal{H}^r_{E,w}, \mathcal{J}^r_E, \mathcal{J}^r_{E,w}$ denote the stratified continuous sheaves of controlled holonomic, weakly controlled holonomic, controlled formal and weakly controlled formal sections of P as in Section 2.4. Also, let $\operatorname{Strat} \mathcal{J}^r$ denote the stratified sheaf given by Definition 5.13. If $P : E \to X$ is a stratified fiber bundle with *manifold fibers*, the sheaf of all stratified $r-$jets will be denoted as $\mathcal{J}^r_0$. (The sheaf $\mathcal{J}^r_0$ is of relevance in the example of a compact Lie group acting on a manifold E with quotient a stratified space X.) For any stratum $L < X$, let E_L denote the induced stratified bundle $P_L : P^{-1}(\overline{L}) \to \overline{L}$. Replacing X by $\overline{L}$, we have induced sheaves $\mathcal{F}|\overline{L}, \mathcal{F}_w|\overline{L}, \mathcal{J}^r_E|\overline{L}, \mathcal{J}^r_{E,w}|\overline{L}, \mathcal{H}^r_E|\overline{L}, \mathcal{H}^r_{E,w}|\overline{L}, \mathcal{J}^r_0|\overline{L}, \operatorname{Strat} \mathcal{J}^r|\overline{L}$. We shall abuse notation slightly and refer to the stratified sheaf given by the collections of induced sheaves

$$\{\mathcal{F}|\overline{L}, \mathcal{F}_w|\overline{L}, \mathcal{J}^r_E|\overline{L}, \mathcal{J}^r_{E,w}|\overline{L}, \mathcal{H}^r_E|\overline{L}, \mathcal{H}^r_{E,w}|\overline{L}, \mathcal{J}^r_0|\overline{L}, \operatorname{Strat} \mathcal{J}^r|\overline{L} : L < X\}$$

also by $\mathcal{F}, \mathcal{F}_w, \mathcal{J}^r_E, \mathcal{J}^r_{E,w}, \mathcal{H}^r_E, \mathcal{H}^r_{E,w}, \mathcal{J}^r_0, \operatorname{Strat} \mathcal{J}^r$. It will be clear from the context whether we are referring to the sheaf or the stratified sheaf over X. We record the following observation for concreteness:

Observation 6.1. *$\mathcal{F}$ and $\mathcal{F}_w$ are isomorphic to $\mathcal{H}^r$ and $\mathcal{H}^r_w$ respectively.*

The morphism from $\mathcal{F}$ to $\mathcal{H}^r$ is obtained by adjoining $r-$jets of holonomic sections, and that from $\mathcal{H}^r$ to $\mathcal{F}$ forgets the decoration.

Definition 6.2. *If X is a manifold, a* differential relation $\mathcal{O}$ (of order r) *is a subsheaf of $\mathcal{J}^r_0$.*

For a stratified space X, a stratified differential relation $\{\mathcal{O}_{\overline{L}} : L < X\}$ *(of order r) is a stratified subsheaf of* $\operatorname{Strat} \mathcal{J}^r$.

We shall need an auxiliary combinatorial organizational tool.

Definition 6.3. *Let (X,Σ), (Y,Σ') be abstractly stratified spaces. A* configuration of indexing sets*, or simply, a configuration, is a set-map $\mathfrak{c}:\Sigma\to\Sigma'$ between the indexing sets Σ,Σ'. We shall say that a stratum-preserving map $f:X\to Y$ is of configuration $\mathfrak{c}$ if $f(S)\subset\mathfrak{c}(S)$ for all $S\in\Sigma$.*

Let (X,Σ), (E,Σ') be the stratifications of X,E. We assume, henceforth, in this subsection, that a configuration $\mathfrak{c}:\Sigma\to\Sigma'$ is fixed, and that the sheaves $\mathcal{F}$, $\mathcal{F}_w$ are *implicitly decorated with an arbitrary, but fixed configuration* $\mathfrak{c}$. For E a manifold, G a compact group, and $X=E/G$, the configuration $\mathfrak{c}$ (used to determine $\mathcal{F}$ or $\mathcal{F}_w$) is uniquely determined by P. This is because the fibers of $P:E\to X$ are manifolds, in particular fibers have a single stratum. Hence, $\mathfrak{c}:\Sigma\to\Sigma'$ is automatically fixed.

Theorem 6.4. *With the assumptions above, the stratified sheaf $\mathcal{F}$ is flexible. In particular, it satisfies the parametric $h-$principle, i.e. $\mathcal{F}\hookrightarrow(\mathcal{F})^*\simeq\operatorname{Strat}\mathcal{J}^r$ is a weak homotopy equivalence. (Hence, by Observation 6.1 $\mathcal{H}^r$ is flexible).*

The proof we give below also shows, mutatis mutandis, that $\mathcal{F}_w$ is flexible. We first prove $\mathcal{F}$ is stratumwise flexible. We begin by proving two general lemmas pertaining to flexibility.

Lemma 6.5. *Let F be a fixed quasitopological space. Let* $\operatorname{Maps}(-,F)$ *be the sheaf over a locally compact topological space X given by*

$$\operatorname{Maps}(-,F)(U):=\operatorname{Maps}(U,F).$$

Then, $\operatorname{Maps}(-,F)$ *is flexible.*

Proof. Let $K_1\subset K_2$ be a pair of compact subsets of X, and W be a topological space. Suppose $\phi:W\times I\to\operatorname{Maps}(-,F)(K_1)$ is a homotopy with a given initial lift $\psi_0:W\times\{0\}\to\operatorname{Maps}(-,F)(K_2)$ of $\phi|W\times\{0\}$. By local compactness of X,

$$\begin{aligned}\operatorname{Maps}(-,F)(K_i)&=\varinjlim_{U\supset K_i}\operatorname{Maps}(U,F_i)\\&=\varinjlim_{U\supset K'}\varinjlim_{K'\supset K_i}\operatorname{Maps}(U,F_i)\\&=\varinjlim_{K'\supset K_i}\varinjlim_{U\supset K'}\operatorname{Maps}(U,F_i)\\&=\varinjlim_{K'\supset K_i}\operatorname{Maps}(K',F_i)\end{aligned}$$

where K' varies over compact neighborhoods of $K_i, i=1,2$. Therefore, we may find compact neighborhoods $K_1'\supset K_1$, $K_2'\supset K_2$ such that ϕ factors through $\phi':W\times I\to\operatorname{Maps}(K_1',F)$ and ψ_0 factors through $\psi_0':W\times\{0\}\to\operatorname{Maps}(K_2',F)$. By further shrinking K_1',K_2' if necessary we may ensure $K_1'\subset K_2'$ and ψ_0' is a lift of $\phi'|W\times\{0\}$ to K_2'.
Since $K_1'\subset K_2'$ is a compact inclusion and hence a cofibration, by [May99, pg. 50], $\operatorname{Maps}(K_2',F)\to\operatorname{Maps}(K_1',F)$ is a fibration. Therefore, ϕ' admits a lift $\psi':W\times I\to\operatorname{Maps}(K_2',F)$ such that $\psi'|W\times\{0\}=\psi_0'$. Let $\psi:W\times I\to\operatorname{Maps}(-,F)(K_2)$ denote the germ of ψ' around K_2. Then $\psi:W\times I\to\operatorname{Maps}(-,F)(K_2)$ is a lift of ϕ, with $\psi|W\times 0=\psi_0$. This proves $\operatorname{Maps}(-,F)(K_2)\to\operatorname{Maps}(-,F)(K_1)$ is a fibration, establishing flexibility of $\operatorname{Maps}(-,F)$. □

Lemma 6.6. *Let $\mathcal{F},\mathcal{G}$ be flexible sheaves on a locally compact topological space X. Then $\mathcal{F}\times\mathcal{G}$ is flexible.*

Proof. Let $K_1 \subset K_2$ be a pair of compact subsets of X. By hypothesis, the restriction maps $\mathcal{F}(K_2) \to \mathcal{F}(K_1)$ and $\mathcal{G}(K_2) \to \mathcal{G}(K_1)$ are fibrations. Therefore, as a product of fibrations is a fibration,

$$(\mathcal{F} \times \mathcal{G})(K_2) = \mathcal{F}(K_2) \times \mathcal{G}(K_2) \to \mathcal{F}(K_1) \times \mathcal{G}(K_1) = (\mathcal{F} \times \mathcal{G})(K_1)$$

is a fibration. This proves the lemma. □

Corollary 6.7. *$\mathcal{F}$ is stratumwise flexible.*

Proof. Since each stratum S is a manifold, and $E|_S$ is a genuine smooth bundle, let F_S denote the fiber of $E|S$. Note that $\mathcal{F}_S = i^*\mathcal{F}_{\bar{S}}$ is the sheaf of sections of $E|S$ over S. Given any $x \in S$ we may choose an open neighborhood U around x over which U trivializes. Therefore, $i_U^*\mathcal{F}_S \cong i_U^* \operatorname{Maps}(-, F_S)$. Hence, $\mathcal{F}_S$ is locally flexible by Lemma 6.5. Global flexibility now follows from Gromov's localization lemma [Gro86, p. 79].

Alternately, partition of unity directly allows us to glue families of sections over a family of sets to extend families of sections, and hence establish that $\mathcal{F}_S$ is flexible. □

Proof of Theorem 6.4. Let $x \in X$ be a point, and S be the unique stratum containing x. Let us choose a neighborhood $W \subset X$ of x such that $W \cong V \times cA$, where $V = W \cap S$ and cA is the normal cone of S in X. Then $\mathcal{F}(W)$ is the quasitopological space of sections of E over W. We obtain a map

$$\operatorname{res}_V^W : \mathcal{F}(W) \to \mathcal{F}_S(V)$$

by restricting a section of E over W to a section of $E|_S$ over $V \subset S$. Explicitly, let $s : V \times cA \cong W \to E$ be a section in $\mathcal{F}(W)$. Let $\widetilde{S}$ be the unique stratum of E containing $s(V)$, and cB be the normal cone of $\widetilde{S}$ in E. Then by the local structure of stratified bundles, $s(v, a) = (t(v), f(v, a))$ for all $(v, a) \in V \times cA$ where $t : V \to \widetilde{S}$ is a section and $f : V \to \Gamma(cA, cB)$ is a $V-$parametrized family of sections of E over cA. The map above is given by $\operatorname{res}_V^W(s) = t$. Consequently, res_V^W is equivalent to the following product fibration, given by projection to the first factor

$$\mathcal{F}_S(V) \times \operatorname{Maps}(V, \Gamma(cA, E)) \to \mathcal{F}_S(V).$$

As a corollary, we obtain $\mathcal{F}(W) \cong \mathcal{F}(V) \times \operatorname{Maps}(V, \Gamma(cA, E))$. As this isomorphism is natural under restrictions to open subsets $W' \subset W$, $V' = W' \cap S \subset V$, it establishes an isomorphism of sheaves

$$i_V^*\mathcal{F} \cong i_V^* \operatorname{Maps}(-, \Gamma(cA, E)) \times i_W^*\mathcal{F}_S$$

By Lemma 6.5, $\operatorname{Maps}(-, \Gamma(cA, E))$ is flexible and by Corollary 6.7, $\mathcal{F}_S$ is flexible. As restriction and products of flexible sheaves are flexible, we obtain $i_V^*\mathcal{F}$ is flexible. Therefore, $\mathcal{F}$ is locally flexible and hence by Gromov's localization lemma [Gro86, p. 79], $\mathcal{F}$ is flexible. □

A description of $\mathcal{H}_S^L$:
Let $S < L$ denote strata of X. Let $\mathcal{H}_S^L$ denote the restriction of

$$\overline{\mathcal{H}}_S^L := \operatorname{hofib}(i_{\overline{S}}^*\mathcal{F}_{\overline{L}} \to \mathcal{F}_{\overline{S}})$$

to the topmost stratum of definition of $\overline{\mathcal{H}}_S^L$, i.e. to the (open) stratum S. Equivalently,

$$\mathcal{H}_S^L = \operatorname{hofib}(i_S^*\mathcal{F}_{\overline{L}} \to \mathcal{F}_S),$$

where we assume that a section $\psi_S \in \mathcal{F}_S(S)$ has been fixed, and homotopy fibers are computed with respect to ψ_S.

Let $U \subset S$ be a local (Euclidean) chart. Note that a small normal neighborhood of U in $\overline{L}$ is of the form $U \times cA_S^L$, where A_S^L is the link of S in $\overline{L}$. Then a neighborhood N_{SL} of S in $\overline{L}$ is an A_S^L-bundle over S. Let S' be the unique stratum in E containing $\psi(S)$. Let $B_{S'}$ be the link of S' in E. Then $\mathcal{H}_S^L(U) = \mathcal{H}_S^L(U, \psi_S)$ consists of two components:

(1) A section of E over $N_{SL}(U)$ restricting to $\psi_S|_U$ over the zero section $U \subset N_{SL}(U)$. Germinally, this is equivalent to a map from U to $\Gamma_c(cA_S^L, cB_{S'})$ as in the proof of Corollary 6.7 above. Note that by Lemma 6.5, the sheaf $\mathrm{Maps}(-, \Gamma_c(cA_S^L, cB_{S'})$ is flexible. We shall refer to this as the *germinal $L-$component.*

(2) A path of sections over U in $\mathcal{F}_S$ starting at $\psi_S|U$, i.e. a continuous map $h : [0,1] \to \mathcal{F}_S(U)$, such that $h(0) = \psi_S|U$. Let $P_\psi(U)$ denote the collection of such maps.

Let $\mathcal{G}$ be a sheaf on S defined by

$$\mathcal{G}(U) = P_\psi(U).$$

Lemma 6.8. *$\mathcal{G}$ is flexible.*

Proof. We note that for U a local chart, the restriction $\mathcal{G}_U$ of $\mathcal{G}$ to U is given by

$$\mathcal{G}_U(V) = \mathrm{Maps}\left((I \times V, \{0\} \times V), (F_S, \psi_S)\right).$$

The homotopy extension property from subcomplexes of S to the fiber F_S then gives the lemma.

Alternatively, we may use Lemma 3.13 to deduce the lemma. □

Using the proof of Theorem 6.4, we can explicitly compute the homotopy fiber $\mathcal{H}_S^L$ for the sheaf $\mathcal{F}$ of sections of $P : E \to X$. Indeed, observe that for quasitopological spaces X, Y, the homotopy fiber of the product fibration $X \times Y \to Y$ over a point $y \in Y$ is *homeomorphic* to $X \times P_y Y$, where $P_y Y \subset \mathrm{Maps}(I, Y)$ consists of the collection of maps $\gamma : I \to Y$ with $\gamma(0) = y$, with the inherited quasitopology. Therefore,

$$\mathcal{H}_S^L \cong \mathrm{Maps}(-, \Gamma_c(cA_S^L, cB_{S'})) \times \mathcal{G}.$$

Proposition 6.9. *$\mathcal{H}_S^L$ is flexible.*

Proof. By Lemma 6.5, $\mathrm{Maps}(-, \Gamma_c(cA_S^L, cB_{S'}))$ is flexible. By Lemma 6.8, $\mathcal{G}$ is flexible. Therefore, using Lemma 6.6, we conclude $\mathcal{H}_S^L$ is flexible. □

Recall Gromov's convention [Gro86, Section 1.4.1] of referring to an arbitrarily small but non-specified neighborhood of a set $K \subset X$ by $\mathrm{Op}\, K$. The following are direct adaptations of Gromov's definitions of the smooth h-principle for manifolds from [Gro86, p. 37] to the stratified context. We spell these out for completeness.

Definition 6.10. *A stratified differential relation $\mathcal{R}$ is said to satisfy the*

(1) stratified h-principle near a subset $K \subset X$ *if for every section $\phi : U(K) \to \mathcal{R}$ on a neighborhood $U(K)$ of $\overline{K}$, there exists an open neighborhood U' of $\overline{K}$, such that $\phi|_{U'}$ is homotopic to a a holonomic section.*

(2) stratified parametric h-principle *near* K *if the map*

$$f \to J^r_f$$

from the space of solutions of $\mathcal{R}$ *on* $\operatorname{Op} K$ *to the space of sections* $\operatorname{Op} K \to \mathcal{R}$ *is a weak homotopy equivalence.*

(3) stratified h-principle for extensions of $\mathcal{R}$, from K_1 to $K_2 \supset K_1$ *if for every section* $\phi_0 : \operatorname{Op} K_2 \to \mathcal{R}$ *which is holonomic on* K_1*, there exists a homotopy to a holonomic section* ϕ_1 *by a homotopy of sections* $\phi_t :, \operatorname{Op} K_2 \to \mathcal{R}$*,* $t \in [0,1]$*, such that* $\phi_t|\operatorname{Op} K_2$ *is constant in* t*.*

(4) parametric stratified h-principle for extensions of $\mathcal{R}$, from K_1 to a $K_2 \supset K_1$ *if the map* $f \to J^r_f$ *from the space of solutions of* $\mathcal{R}$ *on* K_2 *to the space of sections* $\operatorname{Op} K_2 \to \mathcal{R}$ *which are holonomic on* K_1*, is a weak homotopy equivalence.*

6.2. **Holonomic approximation for jet sheaves.** We turn now to generalizing the smooth versions of the h-principle due to Eliashberg-Mishachev [EM01, EM02] to stratified spaces. The following is the *holonomic approximation theorem* for smooth bundles over smooth manifolds.

Theorem 6.11. [EM01, Theorem 1.2.1][EM02, Theorem 3.1.1, p.20] *Let* V *be a manifold,* $E \to V$ *be a smooth bundle, and* $K \subset V$ *be a polyhedron of positive codimension. Let* $f \in \mathcal{J}^r_E(\operatorname{Op} K)$ *be a formal section. Then for any* $\varepsilon > 0$*,* $\delta > 0$*, there exist a diffeomorphism* $h : V \to V$ *with*

$$||h - \mathrm{Id}||_{C^0} < \delta,$$

and a holonomic section $\widetilde{f} \in \mathcal{J}^r_E(\operatorname{Op} K)$ *such that*

(1) *the image* $h(K)$ *is contained in the domain of the definition of the section* f*, and*
(2) $||\widetilde{f} - f|\operatorname{Op} h(K)||_{C^0} < \varepsilon$*.*

In fact, h *may be chosen as the time one value of a diffeotopy* $h_t : t \in [0,1]$*, with* h_0 *equal to the identity,* $h_1 = h$*, and for all* $t \in [0,1]$*,*

(1) *the image* $h_t(K)$ *is contained in the domain of the definition of the section* f*, and*
(2) $||h_t - \mathrm{Id}||_{C^0} < \delta$*.*

Recall that diffeotopies are smooth 1-parameter families of diffeomorphisms [Gro86, p. 37].

Definition 6.12. *Let* S *be a stratum of a stratified space* X*, with link* A*, and* N_S *a normal neighborhood of* S *in* X*; hence* N_S *is a* $cA-$*bundle over* S*. Fix local trivializations* $\{U_i\}$ *and compatible local product metrics* (g^i_S, g^i_{cA}) *on* $U_i \times cA \subset N_S$*. A stratified diffeotopy* $\{h_t : t \in [0,1]\}$ *of* X *supported in* N_S *is said to be normally* $\varepsilon-$*small in the* C^r *norm if the following hold.*

(1) $\operatorname{diam}(h_t(s) : t \in [0,1]) < \varepsilon$ *for all* $s \in U_i \times cA \subset N_S$ *and all* i*.*
(2) *Let* $\phi_{(t,x)} : cA(x) \to cA(h_t(x))$ *denote the map induced by* h_t *from the cone* $cA(x)$ *at the point* $x \in S$ *to the cone* $cA(h_t(x))$ *at the point* $h_t(x) \in S$*. We demand that for all strata* J *of* $cA \setminus \{c_A\}$ *and* $y \in J$*, the* C^r-*norms of* $\phi_{(t,x)}$ *are bounded above by* ε *for all* $x \in S$ *and* $t \in [0,1]$*.*

Note that the conclusion of Theorem 6.11 only ensures C^0-closeness in the conclusion. However, Definition 5.12 allows us to introduce the notion of C^r-closeness of formal sections *in the normal direction.*

Definition 6.13. *We shall say that a pair of C^r-regular formal sections ϕ, ψ of $P : E \to X$ over $U \subset X$ are* normally ε C^r-close *if*

1. *The continuous maps $\phi|\operatorname{diag}(U)$ and $\psi|\operatorname{diag}(U)$ are ε-close in the C^0-norm*
2. *For every pair of strata $S < L$ intersecting U, and $S' = S \cap U, L' = L \cap U$, the germs $\phi_{S',n}, \psi_{S',n}$ are C^r-asymptotic (cf. Definition 5.12), in the following sense: for every $\varepsilon_1 > 0$, there exists $\delta > 0$ such that $\phi_{S',n}|N_\delta(S', L')$ and $\psi_{S',n}|N_\delta(S', L')$ are ε_1-close in the C^r-norm.*

Remark 6.14. *If for every stratum S of X intersecting U, $\phi|_S, \psi|_S$ are C^r-close, then ϕ, ψ are normally C^r-close as well. The converse need not be true.*

To extend Theorem 6.11 to stratified spaces, we need some basic differential topology facts about stratified spaces. The following allows us to extend diffeotopies across strata.

Lemma 6.15. *Let $(X, \Sigma, \mathcal{N})$ be an abstractly stratified space and $S \in \Sigma$ be a stratum. Let $h : S \times I \to S$ be a diffeotopy of S supported on a compact set $K \subset S$. Then there exists an extension of h to a stratified diffeotopy $H : X \times I \to X$. If h is C^0-small, the extension H is normally C^r-small for any $r > 0$. Moreover, if h is C^r-small, H is stratumwise C^r-small as well.*

As a prerequisite to the proof and for later use as well, we will state and prove a general result regarding fiber bundles. The main content of this result is a fiber-preserving analogue of the homotopy extension property. To set it up, let $p : E \to X$, $p' : E \to X'$ be fiber bundles with fibers spaces Y and Y', respectively. Fix basepoints $y \in Y$, $y' \in Y'$ and suppose furthermore that p, p' have as their structure groups the groups of basepoint-preserving homeomorphisms $\operatorname{Homeo}(Y, y)$ and $\operatorname{Homeo}(Y', y')$, respectively. Let $s : X \to E, s' : X' \to E'$ denote the canonical sections of p and p' parametrizing the fiberwise basepoints. Note that $p \times \mathrm{id} : E \times I \to X \times I$ is also a fiber bundle with fiber space Y and structure group $\operatorname{Homeo}(Y, y)$, with a canonical section $s \times \mathrm{id} : X \times I \to E \times I$ traced out by the fiberwise basepoints, as before. Further, suppose X, X', E, E' are equipped with metrics compatible with their topology.

Lemma 6.16. *Suppose $f : X \to X'$ and $g : E \to E'$ are maps such that*

1. *g covers f, i.e. $p' \circ g = f \circ p$, and*
2. *g preserves the sections s, s', i.e. $g \circ s = s' \circ f$.*

Let $F : X \times I \to X'$ be a homotopy such that $F|X \times \{0\} = f$. Then, there exists a map $G : E \times I \to E'$ such that

1. *G covers F, i.e. $p \circ F = G \circ (p \times \mathrm{id})$, and*
2. *G preserves the sections $s \times \mathrm{id}, s'$, i.e. $G \circ (s \times \mathrm{id}) = s' \circ F$.*

Moreover, if $\operatorname{diam}(F(\{x\} \times I)) < \varepsilon$ uniformly for all $x \in X$, then we may choose G such that $\operatorname{diam}(G(\{e\} \times I)) < \varepsilon$ uniformly for all $e \in E$.

Proof. Consider the fiber bundle F^*E' over $X \times I$. This is a principal $\operatorname{Homeo}(Y', y')$-bundle over $X \times I$ which restricts to f^*E' over $X \times \{0\}$, therefore there is an isomorphism of principal $\operatorname{Homeo}(Y', y')$-bundles $(f^*E') \times I \to F^*E'$ over $X \times I$.

The map $g : E \to E'$ covering $f : X \to X'$ furnishes a fiber-preserving map $h : E \to f^*E'$ covering the identity map on X. We collect all these maps in the following commutative diagram:

$$\begin{array}{ccccccc} E \times I & \xrightarrow{h\times \mathrm{id}} & (f^*E') \times I & \xrightarrow{\cong} & F^*E' & \longrightarrow & E' \\ {\scriptstyle p\times \mathrm{id}}\downarrow & & \downarrow & & \downarrow & & {\scriptstyle p'}\downarrow \\ X \times I & = & X \times I & = & X \times I & \xrightarrow{F} & X' \end{array}$$

Let $G : E \times I \to E'$ be the composition of the three horizontal arrows on the top. Then by commutativity of the outer rectangle, G satisfies Condition (1). Next, observe that the leftmost and the rightmost squares satisfy Condition (2). Indeed, the canonical sections of F^*E' and $(f^*E') \times I$ are furnished by the basepoint y' as their structure groups are in $\mathrm{Homeo}(Y', y')$. The middle square is an isomorphism of principal $\mathrm{Homeo}(Y', y')$-bundles; hence it must necessarily preserve the relevant canonical sections. This proves that G satisfies Condition (2) as well.

For the final assertion, observe that as $F(\cdot, t)$ stays uniformly ε-close to f for all $t \in I$, the cocycles of F^*E' are ε-close to those of $(f^*E') \times I$. By the proof of Proposition 1.7 in [Hat17, p. 20], we see that this implies that the fiber-preserving homeomorphism $(f^*E') \times I \to F^*E'$ of the top horizontal arrow in the middle square can be chosen to be uniformly ε-close to identity, where both sides are consider as metric subspaces of $E' \times X \times I$. Thus, the composition of the last two top horizontal arrows $(f^*E') \times I \to E'$ is uniformly ε-close to the map induced by the constant homotopy $X \times I \to X'$, $(x, t) \mapsto f(x)$. The image of $\{z\} \times I \subset (f^*E') \times I \to E'$ under the latter has diameter 0. Therefore, under the former, it has diameter uniformly bounded by ε. This shows that the composition $G : E \times I \to E'$ satisfies the desired property $\operatorname{diam} G(\{e\} \times I) < \varepsilon$ for all $e \in E$. □

Proof of Lemma 6.15. Let A denote the link of S in X. A tubular neighborhood N_S of S in X is a cA-bundle over S, i.e. a fiber bundle over S with fiber cA (cf. Corollary 2.18). The structure group of this bundle is $\mathrm{Homeo}(cA, \{c_A\})$ where $\{c_A\} \subset cA$ is the cone point. By Lemma 6.16, we can extend $h : S \times I \to S$ to a homotopy $f : N_S \times I \to N_S$. Moreover, since $h(\cdot, t)$ is a diffeomorphism, $f(\cdot, t)$ is an isomorphism of bundles, for all $t \in I$.

Next, we construct a stratified diffeotopy $\tilde{h} : N_S \times [0, 1] \to N_S$ by defining $\tilde{h}(x, t) = f(x, 3t)$ for $(x, t) \in N_S \times [0, 1/3]$, $\tilde{h}(x, t) = f(x, 2 - 3t)$ for $(x, t) \in N_S \times [1/3, 2/3]$, and $\tilde{h}(x, t) = x$ for $(x, t) \in N_S \times [2/3, 1]$, smoothing at $N_S \times \{1/3\}$ and $N_S \times \{2/3\}$ if required. We extend to a diffeotopy $H : X \times I \to X$ by defining $H(x, t) = x$ for all $x \in X \setminus K$ and $t \in [0, 1]$.

We now prove the second assertion. Note first that N_S is equipped with a continuous metric that is stratumwise smooth. We can change the metric to an equivalent metric that is locally a product metric on $N_S|K$ thinking of N_S as a cA bundle over S to satisfy the conditions of Definition 6.12. Then, locally on any $U \times cA$ we extend h by the identity on the second coordinate. The resulting extension is then normally trivial, in particular, normally C^r-small on $N_S|K$. Completing the extension to a diffeotopy $H : X \times I \to X$ as above, we see that H is normally C^r-small for any $r > 0$.

Finally note that for any stratum L such that $S < L$, the C^r-distance of $H(\cdot, t)$ and id on L is comparable to the sum of the C^r-distance restricted to S and the normal C^r-distance. The last assertion follows. □

Combining Theorem 6.11 and Lemma 6.15 we have:

Corollary 6.17. *Let X, S be as in Lemma 6.15 above, $P : E \to X$ be a stratified bundle, and $K \subset X$ be a substratified space of positive codimension. Let A denote the link of S and N_S a normal neighborhood of S given by a $cA-$bundle over S. Let $K_S = K \cap S$, and let $N(K_S)$ denote the restriction of the bundle N_S to K_S. Let $f \in \operatorname{Strat} \mathcal{J}^r_E(\operatorname{Op} K)$ be a C^r-regular formal section. Then for any $\varepsilon > 0, \delta > 0$, there exist*

1. *a diffeotopy $h_t : S \to S$ with $h = h_1$ and $||h_t - \operatorname{Id}||_{C^0} < \delta$ for all $t \in [0, 1]$,*
2. *an extension $H_t : X \to X$ of h_t supported in a small neighborhood of $N(K_S)$,*

and a holonomic section $\widetilde{f} \in \operatorname{Strat} \mathcal{J}^r_E(\operatorname{Op} K_S)$ such that

1. *the image $h(K_S)$ is contained in the domain of the definition of the section f,*
2. *$||\widetilde{f} - f|\operatorname{Op}_X h(K_S)||_{C^0} < \varepsilon$,*
3. *$\widetilde{f}$ and $f|\operatorname{Op}_X h(K_S)$ are normally C^r-close on N_S.*

Proof. The existence of a diffeotopy $h_t : S \to S$ with $||h_t - \operatorname{Id}||_{C^0} < \delta$ such that

1. for all $t \in [0, 1]$, the image $h_t(K_S)$ is contained in the domain of the definition of the section f, and
2. there exists a holonomic section $\widetilde{f_S} : \operatorname{Op}_S h(K_S) \to E$ such that $||\widetilde{f_S} - f|\operatorname{Op}_S h(K_S)||_{C^0} < \varepsilon$,

is guaranteed by Theorem 6.11. Note that so far $\widetilde{f_S}$ is defined only on S, and the neighborhood $\operatorname{Op}_S h(K_S)$ is only taken within S.

It remains to extend $\widetilde{f_S}$ into N_S and extend the neighborhood $\operatorname{Op}_S h(K_S)$ to an open neighborhood $\operatorname{Op}_X h(K_S)$. We first apply Lemma 6.15 to h_t to obtain a stratified diffeotopy H_t such that

1. H_t is supported in a small neighborhood of $N(K_S) \subset N_S$.
2. H_t is normally C^r-small.

Let $H = H_1$. Then the first item above guarantees that H is defined and possibly unequal to the identity on $N(K_S)$. Let $cA(x)$ denote the normal cone of S in X at x. Note that $H(cA(x)) = cA(h(x))$ (in fact, the proof of Lemma 6.15 shows that H may be chosen to be the identity in the normal coordinate).

To extend $\widetilde{f_S}$ into N_S holonomically, it suffices to define a holonomic extension of $\widetilde{f_S}$ on the restriction of N_S to K_S. By Remark 5.11, the normal formal section $f_{S,n}$ associated to f is a holonomic section of E over N_S. Our main strategy here is to "graft the conical component of $f_{S,n}$ with $\widetilde{f_S}$" to build a section $\widetilde{f} : \operatorname{Op}_X h(K_S) \to E$. The associated holonomic stratified r-jet would be the desired element $\widetilde{f} \in \operatorname{Strat} \mathcal{J}^r(\operatorname{Op} K_S)$. We accomplish this in an indirect manner in light of the warning at the end of Remark 5.11.

As $\widetilde{f_S}$ and $f|S$ are ε-close as formal r-jets, the sections $\widetilde{f}|_S$ and $f_{S,n}$ from $\operatorname{Op}_S(K) \to E$ are $\varepsilon-$close as well. Let $\widetilde{S}$ denote the unique stratum containing the image of S under $\widetilde{f}|_S$ and $f_{S,n}$. (The existence of such an $\widetilde{S}$ is guaranteed if $\varepsilon > 0$ is sufficiently small.) Let us moreover fix an open neighborhood $U \subset S$ of K_S in S which is contained in the domains of definition of both $\widetilde{f}|_S$ and $f_{S,n}$. Let $\sigma : U \times I \to \widetilde{S}$ be a homotopy through sections $\sigma_t : U \to \widetilde{S}$, between $\sigma_0 = f_{S,n}$ and

$\sigma_1 = \widetilde{f}|_S$. By Lemma 6.16, there exists an extension

$$\widetilde{\sigma} : N_S(U) \times I \to N_{\widetilde{S}} \subset E$$

where $N_{\widetilde{S}}$ is the normal cone bundle of $\widetilde{S}$ in E, and $N_S(U)$ denotes restriction of N_S to U. If $\varepsilon > 0$ is sufficiently small, we may ensure that $h(K_S) \subset U$. Let $\widetilde{f} := \widetilde{\sigma}_1|\operatorname{Op}_X h(K_S)$. Then $\widetilde{f}$ is a holonomic extension of $\widetilde{f}_S|\operatorname{Op}_X h(K_S)$ into a neighborhood of K_S in X. Therefore,

(1) $\widetilde{f}$ is holonomic on $\operatorname{Op}_X h(K_S)$ by construction.
(2) Since f is C^r-regular by hypothesis and equals $\widetilde{f}$ on germs of normal cones on S, f and $\widetilde{f}$ are normally C^r-close on N_S.

This completes the proof. □

It is a well-known fact that for a smooth bundle $P : E \to M$ over a compact manifold M, any two sufficiently close sections $s_1, s_2 : M \to E$ are smoothly isotopic through sections. To see this, fix $s_1(M) = M_1 \subset E$, and let $N_\varepsilon M_1 \subset E$ be a regular normal neighborhood obtained, for instance, by equipping E with a Riemannian metric, and using the exponential map exp to exponentiate from the normal bundle $N_E M_1$ to M_1 down to E. Let H_t denote the linear homotopy on $N_E M_1$ that collapses all the linear fibers down to M_1 (identified with the zero section of $N_E M_1$). If $s_2(M) \subset N_\varepsilon M_1$, then $\exp \circ H_t \circ \exp^{-1}$ gives a smooth isotopy of s_2 to s_1.

The same proof generalizes in a straightforward way to stratified spaces. Let $P : E \to Y$ be a stratified fiber bundle over a compact stratified space Y and $s_1 : Y \to E$ be a stratified section. Let $s_1(Y) = Y_1$. Embedding E in a smooth manifold E' by Theorem 2.8, equipping E' with a Riemannian metric g, and restricting g to E, we obtain a stratumwise Riemannian metric g_s on E. Let $N_\varepsilon Y_1 \subset E$ be a regular normal neighborhood of Y_1 in E obtained by exponentiating the stratumwise normal bundle with respect to the stratumwise Riemannian metric g_s. Then a fiberwise linear homotopy exists as in the manifold case. This establishes the following.

Lemma 6.18. *Let $P : E \to Y$ be a stratified fiber bundle over a compact stratified space Y and $s_1 : Y \to E$ be a smooth stratified section. Further, assume that E is equipped with a metric d_s that is stratumwise smooth Riemannian. Let $s_2 : Y \to E$ be a smooth stratified section. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_s(s_1(y), s_2(y)) < \delta$ for all $y \in Y$, then there exists a stratified isotopy $H : Y \times [0,1] \to E$ such that*

(1) $H(y,0) = s_1(y)$
(2) $H(y,1) = s_1(y)$
(3) $d_s(s_1(y), H(y,t)) < \varepsilon$ *for all* $y \in Y$ *and* $t \in [0,1]$.

Remark 6.19. *Compactness is not essential in the proof of Lemma 6.18. All that was required was the existence of a normal neighborhood as the image of an open neighborhood of the zero-section. This goes through for Y non-compact as well provided we allow the thickness of the open neighborhood to be non-constant.*

As a consequence of Lemma 6.18 we have the following:

Corollary 6.20. *Let $P : E \times (-\varepsilon, 1+\varepsilon) \to Y \times (-\varepsilon, 1+\varepsilon)$ be a stratified fiber bundle over a stratified space $Y \times (-\varepsilon, 1+\varepsilon)$, where Y is a compact stratified space. Let $s_1, s_2 : Y \times (-\varepsilon, 1+\varepsilon) \to E \times (-\varepsilon, 1+\varepsilon)$ be two stratified smooth sections that*

are $\delta-$close in the C^r-norm. Then there exists a section $s_3 : Y \times (-\varepsilon, 1+\varepsilon) \to E \times (-\varepsilon, 1+\varepsilon)$ such that s_3 interpolates smoothly between s_1, s_2, i.e.

1. *$s_3|(-\varepsilon, 0] = s_1|(-\varepsilon, 0]$*
2. *$s_3|[1, 1+\varepsilon) = s_2|[1, 1+\varepsilon)$*
3. *s_3 is $\delta-$close to both s_1, s_2 in the C^r-norm.*

More generally, if there exists a stratum S of Y and a submanifold S' of S such that $s_1|S' \times (-\varepsilon, 1+\varepsilon)$ and $s_2|S' \times (-\varepsilon, 1+\varepsilon)$ are $\delta-$close in the C^r-norm, then there exists a section $s_3 : Y \times (-\varepsilon, 1+\varepsilon) \to E \times (-\varepsilon, 1+\varepsilon)$ such that s_3 interpolates smoothly between s_1, s_2 with C^r-closeness along S', i.e.

1. *$s_3|(-\varepsilon, 0] = s_1|(-\varepsilon, 0]$*
2. *$s_3|[1, 1+\varepsilon) = s_2|[1, 1+\varepsilon)$*
3. *$s_3|S' \times (-\varepsilon, 1+\varepsilon)$ is $\delta-$close to both $s_1|S' \times (-\varepsilon, 1+\varepsilon), s_2|S' \times (-\varepsilon, 1+\varepsilon)$ in the C^r-norm.*

Proof. Let H_t be the homotopy between s_1, s_2 in the discussion preceding Lemma 6.18. Setting $s_3'(x, r) = H_r(x, r)$ for $r \in [0, 1]$ furnishes a linear interpolation. Smoothing slightly at the end-points, e.g. by choosing a smooth monotonically increasing bijective function $g : [0, 1] \to [0, 1]$, and setting $s_3(x, r) = H_r(x, g(r))$ for $r \in [0, 1]$ gives the required s_3. □

We are now in a position to prove the stratified version of Theorem 6.11. All substratified spaces $K \subset X$ below will be assumed to be tamely embedded below, i.e. if K is non-compact, then the closure $\overline{K}$ is a deformation retract of a small regular neighborhood.

Theorem 6.21. *Let X be an abstractly stratified space equipped with a compatible metric, $E \to X$ be a stratified bundle, and $K \subset X$ be a relatively compact stratified subspace of positive codimension. Let $f \in \operatorname{Strat} \mathcal{J}^r_E(\operatorname{Op} K)$ be a C^r-regular formal section. Then for arbitrarily small $\varepsilon > 0$, $\delta > 0$, there exist a stratified diffeomorphism $h : X \to X$ with*

$$||h - \mathrm{Id}||_{C^0} < \delta,$$

and a stratified holonomic section $\widetilde{f} \in \operatorname{Strat} \mathcal{J}^r_E(\operatorname{Op} K)$ such that

1. *the image $h(K)$ is contained in the domain of definition of the section f,*
2. *$||\widetilde{f} - f|\operatorname{Op} h(K)||_{C^0} < \varepsilon$.*
3. *$\widetilde{f}, f|\operatorname{Op} h(K)$ are normally ε C^r-close.*

The same applies for $\operatorname{Strat} \mathcal{J}^r_{E,w}$ in place of $\operatorname{Strat} \mathcal{J}^r_E$.

Proof. The proof proceeds by induction on the depth (cf. Definition 2.1) of X. If X has depth one, it is a manifold, and Theorem 6.11 furnishes the result.

Let S be the lowest stratum (i.e. the stratum of greatest depth) that K intersects. We note that there might be more than one such minimal stratum S_i of possibly varying depths with $K \cap S_i \neq \emptyset$, in which case we shall repeat the argument below for each of these. For convenience of exposition, we assume there is a unique such S. Let $K_S = K \cap S$. Then K_S is compact; else K would intersect a stratum of depth lower than that of S, but there is none such.

Theorem 6.11 ensures the existence of a self-diffeomorphism h_S of S supported in a neighborhood of $K_S = K \cap S$ and a holonomic section $\widetilde{f_S} \in \operatorname{Strat} \mathcal{J}^r_E(\operatorname{Op} h(K_S))$ satisfying the conclusions of the theorem, but only on S. Further, Corollary 6.17 allows us to

(1) extend h_S to a stratified self-diffeomorphism h_e of all of X supported in $N_S \cap \mathrm{Op}_X(K_S)$.
(2) extend $\widetilde{f_S} \in \mathrm{Strat}\, \mathcal{J}^r_E(\mathrm{Op}\, K_S)$ to a stratified holonomic section $\widetilde{f_e} \in \mathrm{Strat}\, \mathcal{J}^r_E(\mathrm{Op}_X\, h(K_S))$ defined on an open neighborhood $\mathrm{Op}_X\, h(K_S)$ of $h(K_S)$ in X.

We may assume without loss of generality that $\mathrm{Op}_X\, h(K_S)$ is the restriction of the normal bundle N_S to $\mathrm{Op}_S\, h(K)$.

Next, delete a (very) small closed neighborhood $N_\eta(S)$ of S in X to obtain X_1, and let $K_1 = K \cap X_1 \subset X_1$. Then the depth of X_1 is strictly less than X, and induction may be applied to obtain

(1) A stratified self-diffeomorphism h_1 of X_1, supported in $\mathrm{Op}_{X_1} K_1$ such that h_1 is $\delta-$close to the identity in the C^0 norm.
(2) a stratified holonomic section $\widetilde{f_1} \in \mathrm{Strat}\, \mathcal{J}^r_E(\mathrm{Op}_{X_1} K)$ defined on an open neighborhood of $h(K_1)$ in X_1.

such that

(1) the image $h_1(K_1)$ is contained in the domain of definition of the section f,
(2) $||\widetilde{f_1} - f|\, \mathrm{Op}\; h_1(K_1)||_{C^0} < \varepsilon$.
(3) $\widetilde{f_1}, f|\, \mathrm{Op}_{X_1}\; h(K_1)$ are normally ε C^r-close on X_1.

Composing h_1 with a further C^0-small stratified diffeomorphism h_2 supported on $\mathcal{A} = N_{2\eta}(S) \setminus \overline{N_\eta(S)}$, we may assume that $h_2 \circ h_1$ and h_e agree on $\mathcal{A}$. Let h be the stratified diffeomorphism obtained by pasting these two diffeomorphisms along $\mathcal{A}$. Note that h is C^0-small, and hence lifts to give a bundle map from $E|\, \mathrm{Op}_X\, K$ to $E|\, \mathrm{Op}_X\, h(K)$ covering $h : \mathrm{Op}_X\, K \to \mathrm{Op}_X\, h(K)$.

Choosing η in $N_\eta(S)$ sufficiently small, we may assume that the domain of the extension $\widetilde{f_e}$ given by the normal bundle to $\mathrm{Op}_S\, h(K)$ constructed earlier in the proof has normal fibers of diameter at least 2η. Thus, on the stratified "annulus" $\mathcal{A}' = (N_{2\eta}(S) \setminus N_\eta(S)) \cap \mathrm{Op}_X\; h(K)$, we have two holonomic sections $\widetilde{f_e}$ and $\widetilde{f_1}$. (We are assuming here that the sections have been composed with the bundle map covering $h : \mathrm{Op}_X\, K \to \mathrm{Op}_X\, h(K)$ described in the previous paragraph.)

Since η is small, and since both $\widetilde{f_e}$ and $\widetilde{f_1}$ are normally C^r-close to f on $\mathcal{A}'$, they are close to each other. Hence, using Corollary 6.20, we can interpolate between the sections $\widetilde{f_e}|N_\eta(S)$ and $\widetilde{f_1}|(X \setminus N_{2\eta}(S))$ to obtain a holonomic section $\widetilde{f}$ on all of $\mathrm{Op}_X\, K$. Since η is small, all three conclusions of the theorem are satisfied by h and $\widetilde{f}$. □

6.3. **Application: Immersions.** A host of examples of open Diff $-$invariant relations have been enumerated by Eliashberg-Mishachev [EM01] in the manifold context and the holonomic approximation theorem deduced for these:

(1) open manifold immersions,
(2) open manifold submersions,
(3) open manifold k-mersions (i.e. mappings of rank at least k),
(4) mappings with nondegenerate higher-order osculating spaces,
(5) mappings transversal to foliations, or more generally, to arbitrary distributions,
(6) construction of generating system of exact differential $k-$forms,
(7) symplectic and contact structures on open manifolds, etc.

Most of these (in particular, immersions, submersions and $k-$mersions of open manifolds) have natural potential generalizations to the stratified context, by replacing the use of Theorem 6.11 by Theorem 6.21.

Stratified Immersions: We illustrate the extra ingredient necessary over and above [EM02, Gro86] by studying the sheaf of stratified immersions in this paper. We postpone a more detailed treatment of applications to subsequent work. Here, we shall prove a stratified analog of the Smale-Hirsch theorem [EM02, Chapter 8.2] (see Theorem 6.24 below).

Definition 6.22. *Let X, Y be abstractly stratified spaces, and let tX and tY denote the tangent microbundles to X and Y respectively. A* stratified immersion *of X into Y is a weakly controlled map $i : X \to Y$ such that the induced map $i_* : tX \to tY$ is a fiberwise stratified embedding of stratified spaces.*

A stratified immersion $i : X \to Y$ will be said to be of *positive codimension*, if the following is true: for any stratum S of X, let S' denote the unique stratum of Y such that $i(S) \subset S'$. Then $i(S) \subset S'$ is an immersed submanifold of positive codimension. Let $\mathrm{Imm}^{\mathfrak{c}}(X, Y)$ denote the quasitopological space of *positive codimension stratified immersions* of a stratified space X in a stratified space Y such that the stratified immersions are of configuration $\mathfrak{c}$.

Given stratified spaces (Y, Σ') and (X, Σ), and a configuration $\mathfrak{c}$ (cf. Definition 6.3), we may construct a sheaf $\mathrm{Imm}^{\mathfrak{c}}(-, Y)$ on X by defining, for each element $U \in \mathrm{Str}(X, \Sigma)$ of the stratified site (Definition 3.6),

$$\mathrm{Imm}^{\mathfrak{c}}(-, Y)(U) = \mathrm{Imm}^{\mathfrak{c}}(U, Y)$$

where U, being an open subset in a stratum-closure of X, is equipped with the induced stratification from X. The restriction maps are defined simply by restriction of the underlying map of an immersion to a smaller subset.

For any stratum $S \in \Sigma$, let $\mathrm{Imm}^{\mathfrak{c}}_{\overline{S}}(-, Y)$ denote the sheaf of positive codimension stratified immersions of elements of the stratified site of the stratified space $(\overline{S}, \overline{S} \cap \Sigma)$ as in Definition 3.8. Let $\mathrm{Imm}^{\mathfrak{c}}_S(-, Y) := i_S^* \, \mathrm{Imm}^{\mathfrak{c}}_{\overline{S}}(-, Y)$ denote its restriction to the open stratum S.

Note that $\mathrm{Imm}^{\mathfrak{c}}(-, Y)$ is a stratified subsheaf of $\mathrm{Maps}(-, Y)$, where $\mathrm{Maps}(-, Y)$ is identified with the sheaf of sections of the surjective map $P : X \times Y \to X$. P is an example of a stratumwise bundle (Definition 2.20), but not a stratified bundle (Example 2.21). Nevertheless, by Remark 4.4, we may describe sections of the Gromov diagonal construction $\mathrm{Maps}(-, Y)^*$ over an element of the site $U \in \mathrm{Str}(X, \Sigma)$ in terms of germs of maps $\sigma : (t(U), U) \to Y$. For every $p \in U$, we may consider $\sigma | t_p U : t_p U \to Y$ as a map $t_p U \to t_{\sigma(p)} Y$. In this process, we can identify sections of $\mathrm{Maps}(-, Y)^*$ over U as microbundle morphisms $tU \to tY$ covering a map $U \to Y$.

Definition 6.23. *A* stratified formal immersion *of X into Y is a pair (F, f) consisting of*

1. *A weakly controlled map $f : X \to Y$,*
2. *A fiber-preserving microbundle morphism $F : tX \to tY$,*

such that F covers f, and F is a fiberwise stratified embedding.

It follows from the discussion above that sections of the Gromov diagonal construction $\mathrm{Imm}^{\mathfrak{c}}(-, Y)^*$ over $U \in \mathrm{Str}(X, \Sigma)$ can be identified with stratified formal

immersions (F, f) of U into Y, where $f : X \to Y$ is of configuration $\mathfrak{c}$. The canonical inclusion $\mathrm{Imm}^{\mathfrak{c}}(-, Y) \hookrightarrow \mathrm{Imm}^{\mathfrak{c}}(-, Y)^*$ over U is given by sending a stratified (holonomic) immersion $i : X \to Y$ to the stratified formal immersion (i_*, i) where $i_* : tX \to tY$ is given by the restriction of $i \times i : X \times X \to Y \times Y$ to the diagonal.

Theorem 6.24. *For any stratified spaces (X, Σ) and (Y, Σ'), and configuration $\mathfrak{c}$ as above, the stratified continuous sheaf $\mathrm{Imm}^{\mathfrak{c}}(-, Y)$ on X satisfies the parametric $h-$principle.*

Proof. By Theorem 4.42, it suffices to check flexibility of $\mathcal{H}^L_S$. Let $S < L$ be strata in X and let $S' = \mathfrak{c}(S), L' = \mathfrak{c}(L)$ be the strata in Y corresponding to S, L through the configuration $\mathfrak{c}$. Let $t_{S,L}$ denote the microtangent bundle to S in $\overline{L}$. Let A_L (resp. $B_{L'}$) denote the link of S (resp. S') in $\overline{L}$ (resp. $\overline{L'}$), and $\{c\} \subset cA_L$ (resp. $\{c'\} \subset cB_{L'}$) denote the cone points of the respective normal cones. Then the microtangent bundle $t_{S,L}$ of S in L is of the form

$$t_{S,L} = TS \oplus N_{S,L},$$

where TS is the tangent bundle to S, and $N_{S,L}$ is a cA_L-bundle over S. We refer to $N_{S,L}$ as the normal cone bundle of S in L. The morphism

$$\mathrm{res}^L_S : i_S^* \mathrm{Imm}^{\mathfrak{c}}_{\overline{L}}(-, Y) \to \mathrm{Imm}^{\mathfrak{c}}_S(-, Y)$$

is given by restricting a germ of a stratified immersion defined on a neighborhood $N_{S,L}$ of S in $\overline{L}$, to the zero section $S \subset N_{S,L}$. Let $U \subset S$ be a chart such that $N_{S,L}$ trivializes as $U \times cA_L$ over U. Then, elements of $i_S^* \mathrm{Imm}^{\mathfrak{c}}_{\overline{L}}(U, Y)$ consist of germs of stratified immersions

$$\varphi : (U \times cA_L, U \times \{c\}) \to Y$$

such that $\varphi | U \times \{c\}$ is an immersion of U in S'. Observe that, for every $p \in U$,

$$\varphi|_{\{p\} \times cA_L} : (cA_L, \{c\}) \to (cB_{L'}, \{c'\}) \subset Y$$

is a germ of a stratified immersion at the cone point c'. Here, the cone $cB_{L'}$ is determined by the configuration $\mathfrak{c}$. In other words, the configuration $\mathfrak{c}$ induces a germ of a stratified embedding at the cone point. Indeed, the microtangent space $t_c(cA_L)$ to the cone cA_L at the cone point c is a germ of a neighborhood of $\{c\}$ in cA_L. The latter is germinally homeomorphic to cA_L itself. Let $\mathrm{Emb}_c(cA_L, cB_{L'})$ denote the quasitopological space of germs of such embeddings. Thus, we have an induced map Φ_φ given by

$$\Phi_\varphi : U \to \mathrm{Emb}_c(cA_L, cB_{L'}),\ \Phi_\varphi(p) = \varphi|_{\{p\} \times cA_L}.$$

Therefore, $\varphi \to (\varphi|_{U \times \{c\}}, \Phi_\varphi)$ furnishes a homeomorphism:

$$i_S^* \mathrm{Imm}^{\mathfrak{c}}_{\overline{L}}(U, Y) \to \mathrm{Imm}(U, S') \times \mathrm{Maps}(U, \mathrm{Emb}_c(cA_L, cB_{L'}))$$

This homeomorphism is natural with respect to passing to smaller open subsets $V \subset U$. It commutes with projections to $\mathrm{Imm}(U, S')$ under restriction maps res^L_S on the left-hand side. It also commutes with the canonical projection to the first factor on the right-hand side. Therefore, we have an isomorphism of continuous sheaves over $U \subset S$,

$$i_S^* \mathrm{Imm}^{\mathfrak{c}}_{\overline{L}}(-, Y) \cong \mathrm{Imm}(-, S') \times i_U^* \mathrm{Maps}(-, \mathrm{Emb}_c(cA_L, cB_{L'})).$$

Under this isomorphism, res^L_S is equivalent to the projection to the first factor of the product sheaf on the right-hand side. Therefore, as in Proposition 6.9,

$$i_U^* \mathcal{H}^L_S \cong P_\psi \mathrm{Imm}(-, S') \times i_U^* \mathrm{Maps}(-, \mathrm{Emb}_c(cA_L, cB_{L'}))$$

By Gromov's Open Extension Theorem [Gro86, p. 86], $\mathrm{Imm}(-, S')$ is a flexible sheaf on S since $\dim(S) < \dim(S')$ by the positive-codimension hypothesis. Thus, the arguments from Proposition 6.9 apply, and we conclude that $\mathcal{H}^L_S$ is a flexible sheaf on S.

Finally, for every stratum S of X, $\mathrm{Imm}^{\mathfrak{c}}_S(-, Y) = \mathrm{Imm}(-, \mathfrak{c}(S))$ is a flexible sheaf on S once again by the Open Extension Theorem [Gro86, p. 86]. Therefore, $\mathrm{Imm}^{\mathfrak{c}}$ is both stratumwise flexible, and infinitesimally flexible across strata. By Theorem 4.42, $\mathrm{Imm}^{\mathfrak{c}}$ satisfies the parametric $h-$principle. □

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